Physics 502: Problem Set 7 (*DUE FRIDAY 4/14*)

1) Magnitude of zero-point energy density.

a) Show that the total density of zero-point energy (i.e., energy per unit volume) associated with the electromagnetic field in vacuum is

$$\frac{2}{L^3}\sum_{\mathbf{k}}\frac{1}{2}\hbar\omega = \frac{\hbar\omega_c^4}{8\pi^2} \tag{1}$$

if one places a cutoff ω_c on the mode frequencies (i.e., assuming modes with higher frequencies either dont exist or, for some reason, dont contribute).

b) Plugging in numbers, estimate the zero-point EM energy stored in our classroom, in Joules, for an energy cutoff of 1 keV (rather mild X-rays). [Keep in mind we have set $c = 1$ everywhere, to get an answer here you will need to restore it!]

2) In class we set up the Lagrangian for the quantized EM field coupled to matter:

$$\begin{aligned} L &= L_{part} + L_{int} + L_{rad}, \\ L_{part} &= \frac{1}{2}m_i\mathbf{v}_i^2 \\ L_{rad} &= \frac{1}{16\pi}\int d^3r\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{8\pi}\int d^3r\,(\mathbf{E}^2 - \mathbf{B}^2) \\ L_{int} &= \int d^3r\,(-\rho\Phi + \mathbf{J}\cdot\mathbf{A}) = \sum_i(-q_i\Phi_i + q_i\mathbf{v}_i\cdot\mathbf{A}_i) \end{aligned} \tag{2}$$

Derive the equations of motion for this Lagrangian and verify that they are given by Maxwell's equations (with sources) and the Lorentz force law.

3) Refer to the "Summary of Electrodynamics (Classical and Quantum)" handout (on the web site under "Handouts"). Please follow the conventions given there, and not those that you might find in some other textbook; in particular, we have $\hat{e}_{\mathbf{k}\lambda} = \hat{e}_{-\mathbf{k}\lambda}$, $\mathbf{Q}^*_{\mathbf{k}\lambda} = \mathbf{Q}_{-\mathbf{k}\lambda}$ and $\mathbf{P}^*_{\mathbf{k}\lambda} = \mathbf{P}_{-\mathbf{k}\lambda}$.

a) In class we expanded $\mathbf{A}$ and its conjugate momentum $\mathbf{\Pi}$ as:

$$\begin{aligned} \mathbf{A} &= \sqrt{\frac{4\pi}{V}}\sum_{\mathbf{k}\lambda} Q_{\mathbf{k}\lambda}\hat{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} \\ \mathbf{\Pi} &= \sqrt{\frac{1}{4\pi V}}\sum_{\mathbf{k}\lambda} P_{-\mathbf{k}\lambda}\hat{e}_{-\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{3}$$

and we showed using $\mathbf{\Pi} = \frac{1}{4\pi}\dot{\mathbf{A}}$ that $P_{-\mathbf{k}\lambda} = \dot{Q}_{\mathbf{k}\lambda}$. Show that $P_{\mathbf{k}\lambda}$ and $Q_{\mathbf{k}\lambda}$ are canonically conjugate, i.e. $P_{\mathbf{k}\lambda} = \frac{\partial L}{\partial \dot{Q}_{\mathbf{k}\lambda}}$.

b) In class, we derived a formula for $H^{(1)}_{int} = \frac{e}{m}\sum_i \mathbf{p}_i \cdot \mathbf{A}_i$ in terms of particle operators $(\mathbf{r}_i, \mathbf{p}_i)$ and photon operators ($a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$). Derive analogous classical and quantum expressions for the Zeeman energy $H_{Zeeman} = -\gamma \mathbf{B}(\mathbf{r_i}) \cdot \mathbf{S}_i$ using $\mathbf{B} = \nabla \times \mathbf{A}$.

c) Give analogous classical and quantum expressions for the total field momentum $\mathbf{P}_{field} = \frac{1}{4\pi}\int d^3r\, \mathbf{E}_T \times \mathbf{B}$. For the classical case, you should find that $\mathbf{P}_{field}$ is proportional to $\sum_{\mathbf{k}\lambda} \mathbf{k} Q_{\mathbf{k}\lambda} P_{\mathbf{k}\lambda}$, and the final quantum result should be $\mathbf{P}_{field} = \sum_{\mathbf{k}\lambda} \hbar \mathbf{k} n_{\mathbf{k}\lambda}$. Discuss the interpretation of this result.

4) Consider a particle of charge q and mass m in a 3-dimensional isotropic simple harmonic oscillator (SHO) potential with frequency ω_0. Let's use the notation that the energy levels are $|n_x n_y n_z\rangle$ with energy $\hbar\omega_0(\frac{3}{2} + n_x + n_y + n_z)$, and consider a transition from initial state $|i\rangle = |001\rangle$ to final state $|f\rangle = |000\rangle$ (i.e., from one of the first excited states to the ground state). Find the rate Γ for spontaneous decay of this system to its ground state via electric-dipole radiation. (Hints: Show that d_{fi} is only along $\hat{z}$ and remind yourself how to evaluate it using raising and lowering operator algebra for the SHO. When evaluating the total emission rate, you can use the usual trick of replacing $(\hat{z} \cdot \hat{e}_{\mathbf{k}\lambda})^2$ by its average value of 1/3.)