Physics 502: Problem Set 4 (*DUE TUESDAY 3/7*)

1) Shankar 19.3.1 p. 533. Note that, as claimed near the top of p. 532, this diverges in the limit of a Coulomb interaction ($r_0 \to \infty$).

2) Shankar 19.3.3 p. 533 (Gaussian potential).

3) Sakurai Chapter 7 Problem 3 (p. 441-442) (finite spherical barrier/well).

4) Let me briefly recap the result derived in class about inelastic scattering in the first Born approximation. The Hamiltonian is $H = H_{\text{probe}} + H_{\text{targ}} + V$. The probe particle is a non-relativistic particle of charge Q and mass M, $H_{\text{probe}} = P^2/2M$, making a transition from $|K_i\rangle$ to $|K_f\rangle$ having energy $\hbar^2 K_i^2/2M$ and $\hbar^2 K_f^2/2M$ respectively. The target system has unperturbed Hamiltonian $H_{\text{targ}} = \sum_j p_j^2/2m + V(\mathbf{r_1}, \mathbf{r_2}, ...)$, and we define the density operator $\rho(r) = \sum_j q_j \delta(\mathbf{r} - \mathbf{r_j})$ and its Fourier transform

$$\tilde{\rho}(\mathbf{K}) = \int d^3\mathbf{r}\, e^{-i\mathbf{K}\cdot\mathbf{r}} \rho(\mathbf{r}) = \sum_j q_j e^{-i\mathbf{K}\cdot\mathbf{r_j}} \tag{1}$$

Finally, the perturbation is the Coulomb interaction,

$$V = \sum_j \frac{Qq_j}{|\mathbf{r}_j - \mathbf{R}|} = \int d^3\mathbf{r} \frac{Q\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{R}|} \tag{2}$$

where $\mathbf{R}$ is the probe coordinate. Then

$$\frac{d\sigma}{d\Omega} = \left(\frac{M}{2\pi\hbar^2}\right)^2 \frac{K_f}{K_i} \left(\frac{4\pi Q}{K^2}\right)^2 |\tilde{\rho}_{ns}(\mathbf{K})|^2 \tag{3}$$

a) Consider the case that $\mathbf{K} = \mathbf{K_f} - \mathbf{K_i}$ is small, and show that the leading approximation for small K is

$$\frac{d\sigma}{d\Omega} = \left(\frac{M}{2\pi\hbar^2}\right)^2 \frac{K_f}{K_i} \left(\frac{4\pi Q}{K^2}\right)^2 |\mathbf{K}\cdot\mathbf{d}_{ns}|^2 \tag{4}$$

where $\mathbf{d}$ is the dipole operator $\int d^3\mathbf{r}\, \mathbf{r}\rho(\mathbf{r})$.

b) Returning to arbitrary $\mathbf{K}$, consider now the case that no transition actually occurs in the target system, so that $n = s$. Show that the differential scattering cross section is what you would expect from *elastic* first-order Born theory for scattering off a static potential given by $U(\mathbf{R}) = \int d^3\mathbf{r} Q\rho_{ss}(\mathbf{r})/|\mathbf{r} - \mathbf{R}|$.

c) Now forget about the special cases of parts (a) and (b). Lets do a case where we can get an exact solution. The target system consists of only a single particle bound to the origin by a 3D simple harmonic oscillator potential, $H_{\rm targ} = p^2/2m + \frac{1}{2}m\omega_0^2 r^2$; we work here with the $|n_x n_y n_z\rangle$ basis based on Cartesian separability (not the one based on spherical Y_l^ms). Obtain the differential scattering cross section $d\sigma/d\Omega$ for a probe particle which enters from the $-\hat{x}$ direction with energy $3\hbar\omega_0$ and exits along the $+\hat{z}$ direction, causing a transition in the target system from ground state $|000\rangle$ to excited final state $|100\rangle$ (i.e., adding one quantum of excitation in the x direction). You may use without proof the following results for certain matrix elements of the one-dimensional SHO: $\langle 0|e^{-ikx}|0\rangle = e^{-k^2/4\alpha}$, $\langle 1|e^{ikx}|0\rangle = (ik/\sqrt{2\alpha})e^{-k^2/4\alpha}$, where $\alpha = m\omega_0/\hbar$. For consistency of solutions, please try to eliminate as many variables as possible (e.g., K_f , K_i, etc.) in favor of ω_0 in your final answer.