Summary of Electrodynamics (Classical and Quantum)

Lagrangian: It is understood that $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\dot{\mathbf{A}}$, $\quad \mathbf{B} = \nabla \times \mathbf{A}$.

$$\mathcal{L}_{\rm part} = \sum_i \tfrac{1}{2} m v_i^2 \, \delta(\mathbf{r} - \mathbf{r}_i)$$

$$\mathcal{L}_{\rm rad} = \frac{1}{8\pi} \left(E^2 - B^2 \right)$$

$$\mathcal{L}_{\rm int} = -\rho\,\Phi + \frac{1}{c}\,\mathbf{J} \cdot \mathbf{A}$$

Hamiltonian: It is understood that $\mathbf{E} = -4\pi c\,\Pi$, $\quad \mathbf{B} = \nabla \times \mathbf{A}$.

$$H = \int d^3r \, \left\{ \frac{1}{8\pi} (E^2 + B^2) + \frac{1}{4\pi}\,\mathbf{E} \cdot \nabla\Phi \right\} + \sum_i \left\{ \frac{1}{2m} \left[\mathbf{p}_i - \frac{q}{c}\,\mathbf{A}_i \right]^2 + q\,\Phi_i \right\}$$

Transverse fields only: It is understood that $\mathbf{E}_{\rm T} = -4\pi c\,\Pi_{\rm T}$, $\quad \mathbf{B} = \nabla \times \mathbf{A}$.

$$H = H_{\rm rad} + H_{\rm part} + H_{\rm int}^{(1)} + H_{\rm int}^{(2)}$$

where

$$H_{\rm rad} = \int d^3r \, \frac{1}{8\pi} \, (E_{\rm T}^2 + B^2) \qquad\qquad H_{\rm part} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{q^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$H_{\rm int}^{(1)} = -\frac{q}{mc} \sum_i \mathbf{p}_i \cdot \mathbf{A}_i \qquad\qquad H_{\rm int}^{(2)} = \frac{q^2}{2mc^2} \sum_i A_i^2$$

Transform to $P_{\mathbf{k}\lambda}$ and $Q_{\mathbf{k}\lambda}$:

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{4\pi c^2} \, \sum_{\mathbf{k}\lambda} Q_{\mathbf{k}\lambda}(t) \, \hat{\mathbf{e}}_{\mathbf{k}\lambda} \, \frac{1}{\sqrt{\Omega}} \, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\Pi_{\rm T}(\mathbf{r}, t) = \frac{1}{\sqrt{4\pi c^2}} \, \sum_{\mathbf{k}\lambda} P_{\mathbf{k}\lambda}(t) \, \hat{\mathbf{e}}_{\mathbf{k}\lambda} \, \frac{1}{\sqrt{\Omega}} \, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

(Continued on other side...)

The classical Hamiltonian pieces become:

$$H_{\text{rad}} = \frac{1}{2} \sum_{k\lambda} \left[P^*_{\mathbf{k}\lambda} P_{\mathbf{k}\lambda} + \omega_k^2 Q^*_{\mathbf{k}\lambda} Q_{\mathbf{k}\lambda} \right] \qquad (\omega_k = ck)$$

$$H^{(1)}_{\text{int}} = -\frac{q}{m} \sqrt{\frac{4\pi}{\Omega}} \sum_i \sum_{k\lambda} \mathbf{p}_i \cdot \hat{\mathbf{e}}_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}_i} Q_{\mathbf{k}\lambda}$$

$$H^{(2)}_{\text{int}} = (\,?\,)$$

Quantize the field amplitudes:

$$Q_{\mathbf{k}\lambda} = \sqrt{\frac{\hbar}{2\omega_k}} (a_{\mathbf{k}\lambda} + a^\dagger_{-\mathbf{k},\lambda})$$

$$P_{\mathbf{k}\lambda} = -i \sqrt{\frac{\hbar\omega_k}{2}} (a_{-\mathbf{k},\lambda} - a^\dagger_{\mathbf{k}\lambda})$$

The quantized Hamiltonian pieces become:

$$H_{\text{rad}} = \sum_{k\lambda} \hbar\omega_k (a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \tfrac{1}{2})$$

$$H^{(1)}_{\text{int}} = -\frac{q}{m} \sum_i \sum_k \left(\frac{2\pi\hbar}{\Omega\omega_k} \right)^{1/2} \hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \mathbf{p}_i e^{i\mathbf{k}\cdot\mathbf{r}_i} (a_{\mathbf{k}\lambda} + a^\dagger_{-\mathbf{k},\lambda})$$

$$H^{(2)}_{\text{int}} = (\,?\,)$$