III. Quantizing EM field

<u>Motivation</u>: So far we have treated electron QM but EM classically

$e \to$ wavefn
$X, P \to$ operators — "semiclassical treatment"
$\vec{A} \to$ number.

Fine when $\vec{A}$ is macroscopic external field
Limitation of <u>weak</u> field.

- Extreme case: hydrogen in free space ($\vec{A}=0$!)
  $|n,l,m\rangle$ is stationary, never decays
  Exptally: $\Gamma \sim 10^9 s^{-1}$
  $\tau \sim 10^{-9} s$
  very unstable!
- Problem? Assuming $\vec{A}=0$!
  True classically
  But QM $\vec{A}$ fluctuates!
  only $\langle \vec{A} \rangle = 0$!

- $\vec{A}(x,t)$ must be promoted to an <u>operator</u>.
  $\to$ First Example of a <u>Qtm field</u>.
  "2nd quantization" $\leftarrow$ Operator that is a fn of $\vec{x}$ & t.
  x treated as parameter here!

- ~~How~~ How to quantize?

→ ~~find~~ take L, find can conj pair

impose $[\ ,\ ]$

find H,

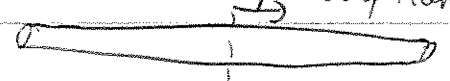
complications: gauge redundancy

$\vec{A}$ vector d.o.f.

- Warmup: "Scalar field"

Consider longitudinal vibrations of 1D rod — 1D version of relativistic KG field $(\partial\phi)^2$

$\eta(x,t)$ displacement @ x



Mass dens ρ
Young modulus Y

$$L = \int dx\, \mathcal{L}(x) = \int dx \left(\frac{1}{2}\rho\dot{\eta}^2 - \frac{1}{2}Y\eta'^2\right)$$

EL eqn: $$\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \eta'} = 0$$

$$\boxed{-\rho\ddot{\eta} + Y\eta'' = 0}$$

wave eqn

$$\eta(x,t) = e^{i(kx-\omega t)} \Rightarrow \rho\omega^2 = Yk^2$$

(relativistic!) dispersion relation

$$\omega = \sqrt{\frac{Y}{\rho}}\,k.$$

Idea: Classically, system = collection of normal modes (SHO)

Quantize each indep'ly ⇒ quantize system

Canonical Quantization (of SHO)

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = \rho \dot{\eta} \quad \text{canon mom.}$$

$$H = \int dx\, \mathcal{H}(x) \qquad \mathcal{H}(x) = \Pi \dot{\eta} - \mathcal{L}$$
$$= \frac{1}{2\rho}\Pi^2 + \frac{1}{2} Y \eta'^2$$

Normal modes: Q_k "coord"

P_k "mom"

$$\begin{cases} \eta(x,t) = \dfrac{1}{\sqrt{\rho L}} \sum_k Q_k(t) e^{ikx} \\ \Pi(x,t) = \sqrt{\dfrac{\rho}{L}} \sum_k P_k(t) e^{-ikx} \end{cases}$$

(check: $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$ $\Downarrow$ $P_k = \frac{\partial L}{\partial \dot{Q}_k}$)

η, Π real $\Rightarrow Q_{-k} = Q_k^*$

$P_{-k} = P_k^*$

$\{Q_k, P_{k'}\} = \delta_{kk'}$ "Poisson Bracket" $\left(\frac{\partial Q}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial x} \right)$

$$H = \int dx \left(\frac{1}{2\rho} \Pi(x,t)^2 + \frac{1}{2} Y \eta'(x,t)^2 \right)$$

$$\frac{1}{2\rho} \cdot \rho \sum_{k,k'} P_k P_{k'} \int \frac{dx}{L} e^{i(k+k')x} = \frac{1}{2} \sum_k P_k P_{-k}$$
$$= \frac{1}{2} \sum |P_k|^2$$

$$\frac{1}{2} Y \cdot \frac{1}{\rho} \sum_{k,k'} Q_k Q_{k'} (ik)(ik') \int \frac{dx}{L} e^{i(k+k')x} = \frac{Y}{2\rho} \sum |Q_k|^2 (-1) k(-k)$$

$$= \frac{1}{2}\sum \omega_k^2 |Q_k|^2$$

So $H = \frac{1}{2}\sum_k \left(P_k^* P_k + \omega_k^2 Q_k^* Q_k\right)$ w/ $\{Q_k, P_{k'}\} = \delta_{kk'}$

See the decomposition into SHO's explicitly!

(now η and $\dot{\eta}$ are playing the role of position & momentum!)

<u>Quantize:</u> Q_k & P_k promoted to qtm <u>operators</u>.

$$\{Q_k, P_{k'}\} \longrightarrow [Q_k, P_{k'}] = i\hbar\,\delta_{kk'}.$$

(twist: $Q_k^\dagger = Q_{-k}$ → not Hermitian!
$P_k^\dagger = P_{-k}$ (ok used to a & $a^\dagger$))

let

$$\boxed{\begin{aligned} a_k &= \sqrt{\frac{\omega_k}{2\hbar}}\left(Q_k + \frac{i}{\omega_k}P_k^\dagger\right) \\ a_k^\dagger &= \sqrt{\frac{\omega_k}{2\hbar}}\left(Q_k^\dagger - \frac{i}{\omega_k}P_k\right) \end{aligned}}$$

$$\boxed{\begin{aligned} Q_k &= \sqrt{\frac{\hbar}{2\omega_k}}\left(a_k + a_{-k}^\dagger\right) \\ P_k &= -i\sqrt{\frac{\hbar\omega_k}{2}}\left(a_{-k} - a_k^\dagger\right) \end{aligned}}$$

$$\boxed{\begin{aligned} [a_k, a_{k'}^\dagger] &= \delta_{kk'} \\ H &= \sum \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right) \end{aligned}}$$

Hilbert space?

vacuum: $|0\rangle$ annihilated by all a_k

$$|\text{state}\rangle = |n_{k_1}, n_{k_2} \ldots\rangle = \prod_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0\rangle$$

$$\omega_k = vk$$

Simplest QFT: 1D scalar field!

Many applications of formalism

- fields are Excitations of real physical system
  "phonons" in solids
  "rotons" & "phonons" in ^4He
  etc.
  CM physics, $v \neq c$, no Lorentz inv.
  But still $a_k, a_k^\dagger$ "quasiparticles"

- fundamental thy of nature
  ~~[illegible]~~
  $v = c$, require Lorentz inv.
  fields represent fundamental particles & interactions
  (Higgs, γ, e, q, ...)

  Usually work w/ $\mathcal{L}$ $\quad L = \int d^4x\, \mathcal{L}$
  Lorentz inv. manifest
  H requires t diff from $\vec{x}$.

Quantization of EM field

Plan:
- Obtain H from $\mathcal{L}$ in terms of normal modes
- canonical qtize
- free field
- field + matter (derive as lifetime of H!)