Physics 417: Problem Set 3 (*Due in class Wednesday 10/2*)

Problem 1: Dirac delta function

Solve *either* **problem 1 or problem 1'. Then treat the other problem as extra credit.**

In class we said that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx} \tag{1}$$

In this problem we will derive this important identity by putting x on an interval and discretizing the momentum. To begin, consider a smooth periodic function $f(x)$ defined on the interval $x \in [-\frac{L}{2}, \frac{L}{2}]$, i.e. it satisfies $f(-\frac{L}{2}) = f(\frac{L}{2})$. A basic fact about such functions is that they can be represented using the *Fourier series*:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \exp\left(\frac{2\pi i n x}{L}\right) \tag{2}$$

for some coefficients f_n called the Fourier coefficients. Now let us define

$$D(x) \equiv \frac{1}{L} \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\pi i n x}{L}\right) \tag{3}$$

We want to show that $D(x)$, in the limit $L \to \infty$, becomes the Dirac delta function.

(a) Show using geometric series (ignoring the fact that it technically doesn't converge) that $D(x) = 0$ if $x \neq 0$.

(b) Show that

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \, f(x) \exp\left(-\frac{2\pi i m x}{L}\right) = f_m \tag{4}$$

for any integer m. (This is the inverse Fourier transform.)

(c) Using the definition of $D(x)$ and the results in (a) and (b), show that

$$\int_{x_1}^{x_2} dx \, D(x) f(x) = f(0) \tag{5}$$

for any interval $[x_1, x_2]$ containing $x = 0$. So in the limit $L \to \infty$, we see that $D(x) \to \delta(x)$.

(d) Finally, by converting the sum over n in (3) into an integral over k in the $L \to \infty$ limit, show that it becomes the desired identity (1).

Problem 1': another derivation of the Dirac delta function identity

Solve *either* **problem 1 or problem 1'. Then treat the other problem as extra credit.**

In this problem we will derive (1) a different way, by regularizing the integral. As we saw in class, the integral is very singular, so let's regularize it by putting in a small exponential damping factor:

$$\widehat{D}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx - \epsilon k^2} \tag{6}$$

We want to compute $\widehat{D}(x)$ for $\epsilon > 0$ where it is well-defined, and then show that as $\epsilon \to 0$, $\widehat{D}(x)$ acquires the defining properties of the delta function.

(a) Perform the Gaussian integral over k and obtain a formula for $\widehat{D}(x)$.

(b) Using your formula in (a), show that as $\epsilon \to 0$, $\widehat{D}(x) \to 0$ for $x \neq 0$.

(c) Now consider the integral

$$I = \int_{x_1}^{x_2} dx\, \widehat{D}(x) f(x) \tag{7}$$

for a general function $f(x)$. Using your previous results, show that as $\epsilon \to 0$, $I \to f(0)$ if $[x_1, x_2]$ contains $x = 0$. (Don't worry about the general properties of $f(x)$ – smoothness, fall-off at infinity, etc. – you can assume whatever is needed.) (You might want to read about the saddle point approximation.)

Problem 2: More properties of the delta function

The delta function is defined by the following properties:

$$\begin{aligned} &\delta(x) = 0 \;\text{ if }\; x \neq 0 \\ &\int_{-\infty}^{\infty} \delta(x) f(x) = f(0) \qquad \text{for any function } f(x) \end{aligned} \tag{8}$$

Use these to prove the following additional properties of the delta function:

(a) $\delta(y) = \delta(-y)$

(b) $f(y)\delta(y-a) = f(a)\delta(y-a)$

(c) $\delta(ay) = |a|^{-1}\delta(y)$ (make sure to check both signs of a!)

(d) $\delta(f(y)) = \sum_i \frac{1}{|f'(y_i)|}\delta(y-y_i)$ provided $f'(y_i) \neq 0$, where the sum is over all y_i satisfying $f(y_i) = 0$ (hint: use (c)!)

(e) Let $\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$ be the step function. Show that $\Theta'(x) = \delta(x)$.

Problem 3: Wavefunctions vs. Dirac notation

Consider a state whose momentum-space wavefunction is:

$$\langle k|\psi\rangle = \begin{cases} 0 & \text{for } k < -k_0/2 \\ N & \text{for } -k_0/2 < k < k_0/2 \\ 0 & \text{for } k > k_0/2 \end{cases} \tag{9}$$

(a) Determine N by requiring that the momentum-space wavefunction is properly normalized.

(b) Determine the position-space wavefunction, $\psi(x) = \langle x|\psi\rangle$. Check that it is normalized correctly by directly integrating over x. (You will need the integral $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} = \pi$.)

(c) Sketch $\langle k|\psi\rangle$ and $\langle x|\psi\rangle$.

Problem 4: Gaussian wavepackets

In class we introduced the Gaussian wavepacket:

$$\psi(x) = \langle x|\psi\rangle = \mathcal{N} \exp\left(-\frac{(x-\bar{x})^2}{4\sigma_x^2}\right) \times \exp(i\bar{k}\,x) \tag{10}$$

where σ_x, $\bar{x}$, and $\bar{k}$ are real numbers and $\mathcal{N}$ is a real normalization. We said it described a state with minimum uncertainty.

(a) Determine $\mathcal{N}$ by requiring that $\psi(x)$ is properly normalized. (The answer is $\mathcal{N} = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_x}}$.)

(b) Choose some values for these parameters and sketch a plot of $\text{Re}(\psi(x))$ and $|\psi(x)|^2$.

(c) Compute $\langle X\rangle$, $\langle P\rangle$, $\langle(\Delta X)^2\rangle$, $\langle(\Delta P)^2\rangle$, and verify that the Heisenberg uncertainty principle is indeed minimized.

(d) Compute the momentum space wavefunction $\langle k|\psi\rangle$ and comment on your result.