Physics 417: Problem Set 2 (*Due in class Wednesday 9/27*)

Problem 1: Some commutator identities

(a) Let A, B, C be arbitrary operators. Verify the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{1}$$

(b) Verify the following identity:

$$[AB, C] = A[B, C] + [A, C]B \tag{2}$$

(c) Use (b) repeatedly in order to derive an analogous identity that relates $[A^n, B]$ to $[A, B]$ (for any integer $n \geq 2$).

(d) Suppose $[A, B]$ commutes with A. Use your result in (c) to show that $[f(A), B] = f'(A)[A, B]$ for any function $f(A)$ defined as a power series in A.

Problem 2: Operator exponentiation

Let's define the *exponential of an operator* via the power series: $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

(a) Prove that $e^A e^{-A} = 1$ for any operator A.

(b) What property should A have so that e^A is unitary?

(c) Remember the spin matrices are given by

$$S_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Derive a simple formula for $e^{i\vec{x}\cdot\vec{S}} \equiv e^{i(x_1 S_1 + x_2 S_2 + x_3 S_3)}$ valid for any complex numbers (x_1, x_2, x_3). (Hint: first calculate $(\vec{x} \cdot \vec{S})^2$.)

(d) What conditions should (x_1, x_2, x_3) satisfy so that $e^{i\vec{x}\cdot\vec{S}}$ is unitary? Check this explicitly using your formula from part (c).

Extra credit: What other important property besides unitarity does $e^{i\vec{x}\cdot\vec{S}}$ have for any $\vec{x}$? How does this follow from the form of the spin matrices (3)?

Problem 3: Compatible vs. incompatible observables

In class we considered a thought experiment where three observables A, B, C were measured and we asked what is the probability P of getting an eigenstate $|c\rangle$ of C starting from an eigenstate $|a\rangle$ of A. If we did not measure B, the answer was $P = |\langle a|c\rangle|^2$ and if we did measure B but summed over all the outcomes, the answer was $P = \sum_b |\langle a|b\rangle|^2 |\langle b|c\rangle|^2$. Show that *if the operators A, B, C have non-degenerate eigenvalues*, then these probabilities are equal if $[A, B] = 0$ *or* $[B, C] = 0$.

Problem 4: More on compatible vs. incompatible Observables

Suppose we have a three state system with orthonormal base kets $|1\rangle$, $|2\rangle$, $|3\rangle$ in which operators A and B are represented as

$$A = \begin{pmatrix} -\frac{a}{2} & \frac{\sqrt{3}a}{2} & 0 \\ \frac{\sqrt{3}a}{2} & \frac{a}{2} & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} \frac{b}{4} & \frac{\sqrt{3}b}{4} & \frac{i\sqrt{3}b}{2} \\ \frac{\sqrt{3}b}{4} & \frac{3b}{4} & -\frac{ib}{2} \\ -\frac{i\sqrt{3}b}{2} & \frac{ib}{2} & 0 \end{pmatrix} \tag{4}$$

for some real numbers a, b.

(a) Find the eigenvalues of A and B individually. Are they degenerate or non-degenerate?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets which are simultaneous eigenkets of A and B. Label them by their eigenvalues under A and B. Does your specification of eigenvalues completely characterize each eigenket?

Problem 5: Uncertainty

(a) Find all linear combinations of $|z+\rangle$ and $|z-\rangle$ eigenkets that maximizes the uncertainty in S_x.

(b) Repeat (a) for S_y.

(c) Repeat (a) for the product of uncertainties $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$.

(d) Verify in (a), (b) and (c) that your results satisfy the uncertainty relation.

Problem 6: Coherent states

The Poisson distribution $P(n;\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$ gives the probability of obtaining n events in a fixed time interval (e.g. the number of emails you receive in a day), assuming the events are independent (uncorrelated) and their probabilities for occurring are the same.

Now consider a Hilbert space spanned by orthonormal states $|n\rangle$, $n = 0, 1, 2, \ldots$ which are eigenstates of the "number operator" N, i.e. $N|n\rangle = n|n\rangle$. The "coherent state" is defined to be

$$|\lambda\rangle \equiv \sum_{n=0}^{\infty} \sqrt{P(n;\lambda)}|n\rangle \tag{5}$$

and it plays an important role in quantum optics. (Roy Glauber got the Nobel prize for this in 2005.)

(a) What is the mean $\langle N\rangle$ and the variance $\langle(\Delta N)^2\rangle$ of the number operator in the coherent state?

(b) Compute the overlap between two coherent states $\langle\lambda'|\lambda\rangle$.

(c) Consider the "annihilation operator" A which acts as $A|n\rangle = \sqrt{n}|n-1\rangle$. Show that $|\lambda\rangle$ is an eigenstate of A and find the eigenvalue. Why does your result not contradict the result of (b)?