# Ensemble Density-Functional Theory for ab-initio Molecular Dynamics of Metals and Finite-Temperature Insulators

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A new method is presented for performing first-principles molecular-dynamics simulations of systems with variable occupancies. We adopt a matrix representation for the one-particle statistical operator $\hat{\Gamma}$, to introduce a "projected" free energy functional $G$ that depends on the Kohn-Sham orbitals only and that is invariant under their unitary transformations. The Liouville equation $[\hat{\Gamma}, \hat{\mathrm{H}}] = 0$ is always satisfied, guaranteeing a very efficient and stable variational minimization algorithm that can be extended to non-conventional entropic formulations or fictitious thermal distributions.

In recent years, the range of problems that can be studied with quantitative accuracy using the methods of computational solid state physics has expanded dramatically. It is now possible to calculate many materials properties with an accuracy that is often comparable to that of experiments. This degree of confidence is based on the fundamental quantum-mechanical treatment offered by density-functional theory (DFT) [1], coupled with the availability of increasingly powerful computers and with the development of algorithms tuned towards optimal performance [2].

The application of these methods and techniques to metallic systems has nonetheless encountered several difficulties, that have made progress slower than for the case of semiconductors and insulators. The discontinuous variation of the orbital occupancies across the Brillouin zone (BZ) makes the occupation numbers rather ill-conditioned variables, and the self-consistent solution of the screening problem can suffer from several instabilities. The absence of a gap in the energy spectrum and the requirement of an exact diagonalization for the Hamiltonian matrix everywhere in the BZ (in order to assign the occupation numbers) introduce "slow frequencies" in the evolution of the orbitals towards the ground state and preclude the straightforward extension to metals of algorithms which performed well for insulators. Smearing the Fermi surface with a finite electronic temperature [3] allows for an improved BZ sampling, but only partially alleviates the problems alluded to above.

In this Letter, we introduce a new approach which solves many of these problems in a natural way, and which provides a general and efficient framework for obtaining the ground state of a Kohn-Sham Hamiltonian at a finite electronic temperature. The typical context is the Mermin formulation for the Fermi-Dirac statistics [4], but the method also applies when generalized entropic functionals are introduced [5], as it is often the case for metallic systems. Other applications include DFT studies of insulators or semiconductors with thermally excited states [6], and fractional quantum Hall states [7]. The language of ensemble-DFT [8] is used, and an orbital-based variational algorithm for the minimization to the ground state is developed and implemented. Dramatic improvements are obtained in the convergence of the energies and especially of the Hellmann-Feynman forces.

Within ensemble-DFT, the Helmholtz free energy functional at a temperature $T$ and for an $N$-representable charge density $n(\mathbf{r})$ in an external potential $V$ is $A_V[n(\mathbf{r})] = F_T[n(\mathbf{r})] + \int V(\mathbf{r})n(\mathbf{r})d\mathbf{r}$, where $F_T$ is the finite-temperature Mermin-Hohenberg-Kohn functional [4]. The charge density $n_0(\mathbf{r})$ that minimizes $A_V$ is the ground-state charge density, and $A_V[n_0(\mathbf{r})]$ is the free energy of the electronic system. A Kohn-Sham-like mapping onto non-interacting electrons leads to a decomposition of the functional $F_T$ into explicit terms (the non-interacting kinetic energy, the classical electrostatical energy, and the entropic contribution) plus the unknown exchange-correlation functional $E_{T,\mathrm{xc}}$, for which we take here the local density approximation [1].

A key assumption is made by adopting a matrix representation $f_{ij}$, in the basis of the orbitals, for the one-particle effective statistical operator $\hat{\Gamma}$; the charge density is correspondingly written as

$$n(\mathbf{r}) = \sum_{ij} f_{ji}\, \psi_i^*(\mathbf{r})\psi_j(\mathbf{r}) \ . \tag{1}$$

Here the $\{\psi_i\}$ are orthonormal single-particle Kohn-Sham orbitals, the sum extends in principle over all the states, and the matrix $f_{ij}$ is constrained to have trace equal to the number of electrons and eigenvalues bounded between 0 and 1. We can then write in all generality the functional $A$ to be minimized as

$$A\,[\,T\,;\{\psi_i\}\,,\,\{f_{ij}\}\,] = \tag{2}$$

$$\sum_{ij} f_{ji}\, \langle\psi_i|\hat{\mathrm{T}} + \hat{\mathrm{V}}_{\mathrm{ext}}\,|\psi_j\rangle \; + \; E_{\mathrm{Hxc}}[n] \; - \; TS[\,\{f_{ij}\}\,] \ ;$$

the Hartree and exchange-correlation terms, which depend on the charge density, have been grouped together. The entropic term is taken to be a function of the eigenvalues of $\mathbf{f}$; in the Fermi-Dirac case, it is $S[\{f_{ij}\}] = \mathrm{tr}\, s(\mathbf{f})$, where $s$ is $f \ln f + (1-f)\ln(1-f)$. In most applications, the external potential $V_{\rm ext}$ is generated by an array of non-local ionic pseudopotentials. The free energy functional defined in Eq. (2) is in the form of traces of operators, and so it is covariant under a change of representation (i.e., for a unitary transformation $\mathbf{U}$ of the orbitals $\{\psi_i\}$); this can be verified by letting $\mathbf{f} \to \mathbf{f}' = \mathbf{U}\,\mathbf{f}\,\mathbf{U}^\dagger$ and $|\psi_j\rangle \to |\psi_j\rangle' = \sum_m U^*_{jm}\psi_m$. The covariance of the free energy functional $A$ allows for the definition of a new, *projected functional* $G$, that depends only on the orbitals $\{\psi_i\}$:

$$G\,[\,T\,;\{\psi_i\}\,] := \min_{\{f_{ij}\}} A\,[\,T\,;\{\psi_i\}\,,\,\{f_{ij}\}\,] \ . \tag{3}$$

$G$ is *invariant* under any unitary transformation of the $\{\psi_i\}$: the transformed orbitals cannot lead to a different value for $G$, by virtue of the covariance of $A$.

The projected functional $G$ represents a much better conditioned choice than the original free energy $A$ when it is used in minimization algorithms that are based on the orbital propagation towards self-consistency, as is the case in the Car-Parrinello or conjugate gradients methods [2]. The reasons are several, albeit related. (1) The functional $G$ no longer depends on the occupancies of the orbitals or on their unitary transformations ("rotations") in the occupied subspace. These are ill-conditioned, non-local degrees of freedom, with the added non-linear constraint of charge normalization. (2) The $f_{ij}$ in this formalism have become dependent variables, implicitly defined by the minimization in Eq. (3), and this dependency does not enter into the calculation of the functional derivatives $\delta G/\delta\psi_i^*$, since the contributions $(\partial G/\partial f_{kl})(\partial f_{kl}/\partial\psi_i^*)$ are zero because of the minimum condition. (3) The occupancies of the orbitals and their rotations in the occupied subspace are now consistently considered as part of the same problem, namely that of finding the ground-state statistical operator, and not—as it is usually done—as two independent degrees of freedom. The evolution of the $f_{ij}$ turn out to be largely decoupled from the problem of updating the orbitals in the subspace orthogonal to the occupied subspace. (4) The expensive and inefficient evolution for the orbital rotations is now shifted to the matrix $\mathbf{f}$; this implies that the associated slow frequencies in the evolution of $A$ have now been compressed to zero by the minimization condition. Subspace alignment [9] between subsequent orbital updates is also automatically enforced.

This formulation naturally separates the evolution of the orbitals $\{\psi_i\}$ from that of the $f_{ij}$: the orbitals get updated in an outer loop that minimizes $G$ to selfconsistency, and after every update an inner loop on the $f_{ij}$ minimizes $A$ *at fixed orbitals* $\{\psi_i\}$. In other words, the $f_{ij}$ are projected via the inner loop onto the $G$ surface, where $\mathbf{f}$ (or $\hat{,}$, of which $\mathbf{f}$ is the matrix representation in the orbital basis) commutes with the non-selfconsistent Hamiltonian. The minimization of $G$ is then freed from all constraints but the orthonormality of the $\{\psi_i\}$, and the rotations of the orbitals do not play any role.

The commuting relationship of $\hat{,}$ and $\hat{\mathrm{H}}$ can be made explicit from the minimum condition. Let us define

$$h_{ij} \;=\; \langle\psi_i|\hat{\mathrm{T}}+\hat{\mathrm{V}}_{\rm ext}|\psi_j\rangle \ \ , \ \ V_{ij}^{[n]} \;=\; \langle\psi_i|\hat{\mathrm{V}}_{\rm Hxc}^{[n]}|\psi_j\rangle$$

for the matrix elements of the Hamiltonian (the superscript $[n]$ is a reminder that the potential $\mathrm{V}_{\rm Hxc}^{[n]}$ depends self-consistently on the charge density); the minimum condition that defines $G$ implies

$$\begin{aligned}\frac{\delta A}{\delta f_{ji}} \;&=\; 0 \;=\; h_{ij} + \frac{\delta E_{\rm Hxc}}{\delta f_{ji}} - T\frac{\delta S}{\delta f_{ji}} - \mu\,\delta_{ij}\\ &=\; h_{ij} + \int d\mathbf{r}\,\frac{\delta E_{\rm Hxc}}{\delta n(\mathbf{r})}\,\frac{\delta n(\mathbf{r})}{\delta f_{ji}} - T\,[s'(\mathbf{f})]_{ij} - \mu\,\delta_{ij}\\ &=\; h_{ij} + V_{ij}^{[n]} - T\,[s'(\mathbf{f})]_{ij} - \mu\,\delta_{ij} \quad . \end{aligned}\tag{4}$$

The constraint of charge conservation $\mathrm{tr}\,\mathbf{f} = N$ is taken into account with the introduction of the Lagrange multiplier $\mu$, and the notation $[s'(\mathbf{f})]_{ij}$ is used in place of $d\,\mathrm{tr}\,s(\mathbf{f})\,/df_{ji}$. The stationary condition in (4) is thus

$$h_{ij} + V_{ij}^{[n]} - T\,[s'(\mathbf{f})]_{ij} \;=\; \mu\,\delta_{ij} \ \ . \tag{5}$$

It follows that $f_{ij}$ and the Hamiltonian $h_{ij} + V_{ij}^{[n]}$ are diagonalized by the same unitary rotation, at fixed orbitals, and thus represent "commuting" operators; the non-selfconsistent Liouville equation $[\hat{,}\,,\hat{\mathrm{H}}] = 0$ is satisfied. The relation (5) does not imply that $h_{ij} + V_{ij}^{[n]}$ and $f_{ij}$ are diagonal, but just that there is a common transformation that diagonalizes both; the formalism per se is not linked to a preferred diagonal representation.

The inner loop for the update of the occupation matrix $f_{ij}$ is carried out at fixed orbitals, and so it does not require the calculation of new matrix elements for the kinetic energy operator or the non-local pseudopotential, and there are no orthogonalizations involved. The Fourier transforms of the $\{\psi_i\}$ can also be eliminated by storing their real-space representation. We have chosen for the $f_{ij}$ an iterative minimization that has a simple and appealing rationale: if the problem were not self-consistent, the solution for the equilibrium $f_{ij}$ would be found by straightforwardly diagonalizing the Hamiltonian matrix, calculating from its eigenvalues the thermal distribution of the occupation numbers, and rotating them back into the current orbital representation. Since the problem is self-consistent, this will not be the actual solution, but we use it as a search direction for a direct line minimization in the multidimensional space of the $f_{ij}$. The procedure is organized as follows. The matrix

$h_{ij}$, with the kinetic-energy and non-local contributions, is determined once for all before entering the inner loop. The updated charge density (let us assume that the $m^{th}$ iteration in the inner loop is taking place) is calculated as

$$n^{(m)}(\mathbf{r}) \;=\; \sum_{ij} f_{ji}^{(m)}\; \psi_i^*(\mathbf{r})\psi_j(\mathbf{r}) \;\;. \tag{6}$$

The Hartree and exchange-correlation energy $E_{\rm Hxc}^{(m)}$ and potential $V_{\rm Hxc}^{(m)}(\mathbf{r})$ are then calculated, and the matrix representation $V_{ij}^{(m)}$ is constructed. The entropy $S^{(m)}$ is also computed, following a diagonalization of $\mathbf{f}$,

$$f_{ij}^{(m)} = \sum_l Y_{il}^{(m)\dagger} f_l^{(m)} Y_{lj}^{(m)} \;\;. \tag{7}$$

Actually, in a traditional plane-wave approach the charge density (6) is calculated more efficiently in this representation in which the $f_{ij}$ are diagonal, since a temporary rotation of the orbitals can then be performed on their more compact reciprocal-space representation. The Hamiltonian matrix is then updated using the new local terms, and diagonalized as

$$H_{ij}^{(m)} = h_{ij} + V_{ij}^{(m)} = \sum_l Z_{il}^{(m)\dagger} \epsilon_l^{(m)} Z_{lj}^{(m)} \;\;. \tag{8}$$

The non-self-consistent minimum for $\mathbf{f}$ would now be

$$\widetilde{f}_{ij}^{(m)} = \sum_l Z_{il}^{(m)\dagger} f_T(\epsilon_l^{(m)} - \mu) Z_{lj}^{(m)} \;\;, \tag{9}$$

where $f_T$ is the (Fermi-Dirac) thermal distribution. We choose this as our search direction in the $f_{ij}$ space, and a full line minimization is performed along the multi-dimensional segment $\mathbf{f}_\beta^{(m+1)} \;=\; \mathbf{f}^{(m)} + \beta\,\Delta\mathbf{f}^{(m)}$, where $\Delta\mathbf{f}^{(m)} = \widetilde{\mathbf{f}}^{(m)} - \mathbf{f}^{(m)}$. Note that $\beta$ parametrizes an *unconstrained* search, since $\mathrm{tr}\,\mathbf{f}_\beta \;=\; N$ at the end-points and thus, by linearity, at all $\beta$. The search direction is determined by the eigenvalues of the non-self-consistent Hamiltonian, attracting the $f_{ij}$ towards the representation in which they commute with the Hamiltonian and towards the thermal equilibrium values that they would assume for this Hamiltonian. Since the search direction is determined by the eigenvalues/eigenvectors of $\hat{\mathrm{H}}$, and not from the occupation numbers, this current formalism can also be applied when generalized entropic functionals are defined, or when non-monotonic thermal distributions are introduced [5].

The direct minimization proceeds by calculating the free energy and its derivative along the search line at the two end-points $\beta = 0$ and $\beta = 1$, *taking into account the self-consistent variations in the charge density* and thus in the Kohn-Sham Hamiltonian. The line derivative $A'$ is $\sum_{ij} \Delta f_{ji}^{(m)}\,(\delta A / \delta f_{ji})$, where

$$\left.\frac{\delta A}{\delta f_{ji}}\right|_{\beta=0} \;=\; \left[\, h_{ij} + V_{ij}^{(m)} - T \sum_l Y_{il}^{(m)\dagger} s'(f_l^{(m)}) Y_{lj}^{(m)} \right] ;$$

$A'(\beta = 0)$ is always smaller than 0, and so the iterative update of $\mathbf{f}$ takes place in a strictly variational fashion. We then calculate the charge density $\widetilde{n}^{(m)}(\mathbf{r})$, the matrix elements $\widetilde{V}_{ij}^{(m)}$, and the free energy $\widetilde{A}$ corresponding to $\beta = 1$, together with the line derivative $A'(\beta = 1)$ via

$$\left.\frac{\delta A}{\delta f_{ji}}\right|_{\beta=1} \;=\; \left[\, h_{ij} + \widetilde{V}_{ij}^{(m)} - T \sum_l Z_{il}^{(m)\dagger} s'(\widetilde{f}_l^{(m)}) Z_{lj}^{(m)} \right] .$$

Since the kinetic energy and the pseudopotential contributions are exactly linear along the search direction (only the prefactors $f_{ij}$ change), and the Hartree energy is quadratic, while the remaining exchange-correlation and entropic terms are very well-behaved, a cubic or a parabolic interpolation locates the value of $\beta$ corresponding to the minimum with very high accuracy. More importantly, the choice of a direct minimization for $\mathbf{f}$ along a linear search implies that *level-crossing instabilities are completely eliminated*, even in the limit of zero temperature and/or in the absence of an entropic term. In practice, we find that one or two iterations in the inner loop are the optimal choice even for large systems (e.g., the diffusion of an adatom on a slab of 144 atoms [10]), since we also need self-consistency with respect to $\{\psi_i\}$.

$G$ can be minimized very efficiently with a direct all-bands conjugate-gradients method; however, since it has a much broader spectrum for its eigenvalues than in an insulating case, it exhibits a slower convergence. This is essentially due to the occupancies of the higher bands being close to zero. To solve this problem, we have resorted to a preconditioning strategy: we choose a set of scaled variables in which the functional has a more compressed spectrum, and the search directions are calculated in this new metric. In the diagonal representation the total energy around the minimum is a quadratic form $\sum_{i,n} f_i \epsilon_i c_{i,n}^2$ ($c_{i,n}$ is the expansion coefficient of $\psi_i$ in the $n$-th element of the basis set); if the steepest-descent directions are chosen according to scaled variables $\widetilde{c}_i \;=\; \sqrt{f_i}\, c_{i,n}$, the preconditioned steepest-descent directions for the original variables become [11]

$$-\frac{\delta G}{\delta \psi_i^*} \;\rightarrow\; -\frac{1}{f_i}\,\frac{\delta G}{\delta \psi_i^*} \;=\; -\,\hat{\mathrm{H}}\,\psi_i \;\;. \tag{10}$$

With some degree of overcorrection [12], these can also be used to construct conjugate directions. A generalization to our case is obtained by calculating the steepest-descent directions $\mathbf{g}_i$ of $G$, passing them into the diagonal representation where they can be preconditioned as in (10), and transforming them back in the initial representation. The steepest-descent directions are

$$\mathbf{g}_i \;=\; -\frac{\delta G}{\delta \psi_i^*} \;=\; -\sum_j f_{ji} \hat{\mathrm{H}} |\psi_j\rangle \;;\; \mathbf{g}'_i \;=\; -f'_{ii} \hat{\mathrm{H}} |\psi'_i\rangle \;, \tag{11}$$

where the primed term refers to the diagonal representation ($\mathbf{f}' = f'_{ii}\,\delta_{ij} = \mathbf{U}\,\mathbf{f}\,\mathbf{U}^\dagger$). The preconditioned steepest descents $\mathbf{G}'_i$ and $\mathbf{G}_i$ are thus

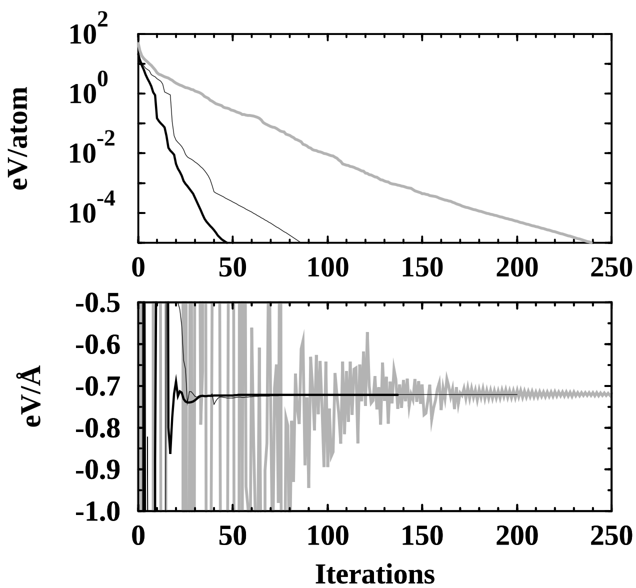


FIG. 1. Convergence of the total free energy (upper panel, semi-logarithmic scale) and of the force on the surface atom (lower panel) in the 15-layer Al(110) slab. Grey line, ABV; thin solid line, ensemble-DFT with 2 iterations in the inner loop; thick solid line, ensemble-DFT with 4 iterations in the inner loop.

$$\mathbf{G}'_i \;=\; -\hat{\mathrm{H}}|\psi'_i\rangle \;=\; -\hat{\mathrm{H}}\left(\sum_m U^*_{im}|\psi_m\rangle\right) \quad , \qquad (12)$$

$$\mathbf{G}_i \;=\; \sum_n U^{*\dagger}_{in}\mathbf{G}'_n \;=\; -\,\hat{\mathrm{H}}|\psi_i\rangle \quad . \qquad (13)$$

Such preconditioned gradients $\mathbf{G}_i$ greatly improve the convergence rate, given that higher bands are now updated with the same speed as lower filled bands, and are much cheaper to compute than the $\mathbf{g}_i$ in Eq. (11). In addition to this occupancy preconditioning, a standard kinetic-energy preconditioning should also be used in plane-wave calculations. One iteration on the orbitals consists thus of several operations: 1) each preconditioned gradient $-\hat{\mathrm{H}}|\psi_i\rangle$ is calculated, conjugated with the previous search direction, and projected out of the subspace spanned by the orbitals to assure first-order orthonormality along the search; 2) the first derivative of the free energy along the multidimensional (all bands, all plane-waves, and all **k**-points) line is calculated, and a trial step along the search line is taken; 3) after reorthogonalizing the orbitals, the new free energy provides the third constraint to identify the optimal, parabolic minimum along the search line.

The complete algorithm provides a remarkably robust and efficient convergence. As a paradigmatic case we present here results for a unit cell that is 32 Å long, and contains a 15-layer 1×1 Al(110) slab. We use the single **k**-point $\frac{2\pi}{a_0}(\frac{1}{4},\frac{1}{4},\frac{1}{4})$, a fictitious Gaussian temperature [13] of 4 eV, and 64 orbitals. The large value of the temperature assures that the coarse sampling is sufficient; similar results are obtained with smaller and more physical temperatures. Fig. 1 monitors the convergence of the total free energy and of the Hellmann-Feynman forces as a function of the number of iterations on the $\{\psi_i\}$; we compare an optimal all-bands variational implementation (ABV) [13] with the scheme that we have described (ensemble-DFT) [14]. The improved convergence for the total energies, and particularly for the Hellmann-Feynman forces, is clearly apparent. It should be stressed that, at variance with the ABV case, the behavior of the total free energy in the line searches in both the outer and the inner loops is accurately parabolic; interpolated minima are usually fractions of a percent off their true values. The improved convergence of the ionic forces leads in particular to a much tighter conservation of the constant of motion in molecular dynamics simulations. We show in Fig. 2 the results of a run for our Al(110) slab. The timestep is 2 fs; the ions are moved after a fixed tolerance in the convergence of the free energy is reached (identical results are obtained if a fixed number of iterations is used). The systematic drift of the constant of motion stabilizes after $\sim 0.3$ ps of thermalization to $-0.6$ eV/cell/ps for the ABV case, and to $-0.0008$ eV/cell/ps for ensemble-DFT. Such stability opens the way to inexpensive molecular dynamics simulation of large metallic systems even on common workstations [10].

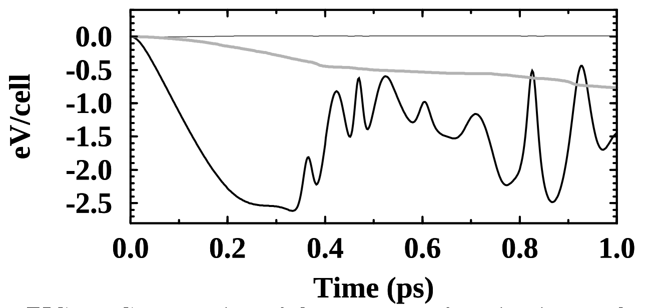


FIG. 2. Conservation of the constant of motion in a molecular-dynamics run for the 15-layer Al(110) slab. The bottom curve is the electronic free energy; the top two curves add to that the kinetic energy of the ions (the grey line is for ABV, the solid thin line is for ensemble-DFT with 2 iterations in the inner loop).

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[1] See e.g. R. O. Jones and O. Gunnarsson, Rev. Mod. Phys. **61**, 689 (1989).
[2] R. Car and M. Parrinello, Phys. Rev. Lett. **55**, 2471 (1985); M. C. Payne, M. P. Teter, D. C. Allan, T. A. Arias, and J. D. Joannopoulos, Rev. Mod. Phys. **64**, 1045 (1992); D. Vanderbilt, Phys. Rev. B **41**, 7892 (1990).
[3] C.-L. Fu and K.-M. Ho, Phys. Rev. B **28**, 5480 (1983); M. J. Gillan, J. Phys.: Condens. Matter **1**, 689 (1989).
[4] N. D. Mermin, Phys. Rev. **137**, A1441 (1965).
[5] A. De Vita, Ph.D. Thesis, University of Keele (1992); N. Marzari *et al.*, to be published.
[6] A. Alavi, J. Kohanoff, M. Parrinello, and D. Frenkel, Phys. Rev. Lett. **73**, 2599 (1994).
[7] O. Heinonen, M. I. Lubin, and M. D. Johnson, Phys. Rev. Lett. **75**, 4110 (1995); M. Ferconi, M. R. Geller, and G. Vignale, Phys. Rev. B **52**, 16357 (1995).
[8] See e.g. R. G. Parr and W. Yang, *Density-Functional Theory of Atoms and Molecules*, Oxford University Press, New York (1989).
[9] T. A. Arias, M. C. Payne, and J. D. Joannopoulos, Phys. Rev. Lett. **69**, 1077 (1992).
[10] N. Marzari, Ph.D. Thesis, University of Cambridge (1996); N. Marzari *et al.*, to be published.
[11] M. J. Gillan, private communication.
[12] The mixing factor used to conjugate a direction depends on the scalar product between the original and the preconditioned gradient; we just use the preconditioned gradient on both sides, without noticing a deterioration in the convergence.
[13] J. M. Holender, M. J. Gillan, M. C. Payne, and A. D. Simpson, Phys. Rev. B **52** 967 (1995).
[14] The cost of one ensemble-DFT iteration (with 2 $f_{ij}$ iterations in the inner loop) ranges between 2/3 and 2 (in ABV units); the extrema are for the Fourier-transform limited regime with or without the $\{\psi_i\}$ stored in real-space. The cost in the cubic-scaling regime is $\sim 3/2$, or lower if the non-local pseudopotentials are in the reciprocal-space representation.