Some notes regarding Kepler's 3rd Law and Newton

Mark Croft & Theo Gallien-Currier

Note: Heuristic work-in-progress for teaching purposes only.

Introduction

The objects of this heuristic work are several-fold. Firstly, to consider reasoning that Kepler followed in determining his 3rd Law. Secondly, to introduce Newton's derivation/generalization of Kepler's 3rd Law in which the relative mass of the central object is fixed by fitting the Law to a system. Thirdly, to fit Newton's version of Kepler's 3rd Law to selected systems (Sun-planets; Jupiter-moons; Saturn-moons; earth-Moon/satellites) thereby illustrating both the universality the $P^2 \sim a^3$ relation and the quantitively estimation of the relative masses of the central objects.

Planetary Orbits/Periods about the Sun

Copernicus was very reasonably proud that his quantitative formulation of the Sun centered solar system had an extremely orderly, and monotonic relation between the period of a planets orbit **P** (in years) and the radius of the planets circular orbit around the Sun **a** (in AU). Figure 1 illustrates this monotonic, albeit quite curved, relation for the planets known to Copernicus. It is important to note that with these units the Earth is by definition at the point P= 1 year and a= 1 AU.

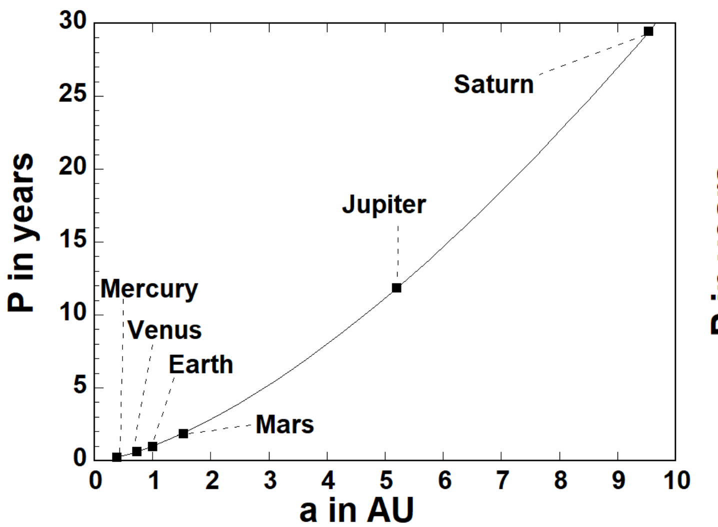


Figure 1a) Monotonic (but curved) relation between the period of a planets orbit **P** (in years) and the distance of the planets orbit around the Sun **a** (in AU). Only the planets known to the ancients are included. For Copernicus the orbits were circular and **a** is the radius. After Kepler (see below) the orbits were known to be elliptical and **a** is semi-major axis (1/2 the long elliptical axis).

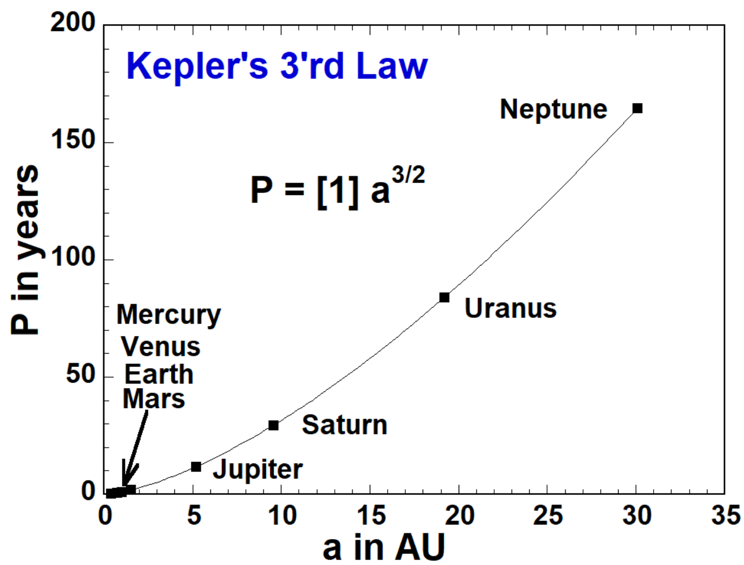


Figure 1b) Monotonic (but curved) relation between the period of a planets orbit **P** (in years) and the distance of the planets orbit around the Sun **a** (in AU). Here all of the current planets are included and the **a** is semi-major axis (1/2 the long elliptical axis). The solid line connecting the data represents Kepler's 3rd Law (see

Kepler's 3rd Law formulation and its logarithmic reformulation

At this juncture the reader is referred to Appendix I, where a good online discussion of how Kepler might have determined his 3rd Law is quoted/paraphrased. The bottom line from this discussion is that: 1) Kepler knew about logarithms; and 2) the natural way to identify power law behavior is by the slope of a log-log plot. Logarithmic plots are common in astronomy where data, over many orders of magnitude (powers of 10), need to be summarized.

Kepler's 3rd law quantified Copernicus's monotonic **P** vs **a** variation with the power law dependence:

$$\mathbf{P}^2 = \mathbf{k}^2\, \mathbf{a}^3 \,. \qquad (1a)$$

Where again **P** is the orbital period of a planet about the Sun (in years) and **a** is the radius (semimajor axis) of the orbit (in Astronomical Units, AU). An equivalent for the 3rd Law follows by taking the square root of both sides of the equation (1):

$$\mathbf{P = k\,a^{3/2}} \quad (1b)$$

In Figure 1b the orbital-period/semimajor-axis for all of the currently known planets is plotted along with the Kepler's 3rd Law variation (solid line). Note how the inner solar system planets on crowded in close to the origin. This is typical when the vertical axis is an exponential power of the horizontal variable causing it to grow rapidly.

Since Kepler appears to have used logarithms in his deduction of his 3rd Law (see Appendix I) it is useful to mobile formulate a logarithmic 3rd Law. Taking the logarithm of both sides equation (1b) and doing rearranging leads to:

$$\mathbf{Log(P) = Log(k\,a^{3/2})} \quad (2a)$$

$$\mathbf{Log(P) = Log(k) + Log(a^{3/2})} \quad (2b)$$

$$\mathbf{Log(P) = Log(k) + 3/2\,Log(a)} \cdot \quad (2c)$$

Thus, the slope of the Log(P)-Log(a) plot is the exponential in the power law, 3/2.

Application of Kepler's 3rd Law (logarithmic formulation) to the Sun-planets

Figure 2a is a Log-Log plot of the same the orbital-period/semimajor-axis data from the previous figure. Note how the inner and outer solar system points can all be clearly seen. Here it is important to note that the scales in the graph are logarithmic so that you don't actually have to look-up/compute the logarithms of the numbers. As an old man, I can well remember Log-Log graph paper that my engineer-father first introduced me to. In Figure 2a a linear (straight-line) dependence of the data is abundantly clear. The formula shown in the figure is a "best-fit" of the data to a straight line. The slope of the line determines the power law. Note also the fit value at P=1 and a=1 yields k very very close 1 in formulas 1-2.

Figure 2b is a Log-Log plot of the orbital-period/semimajor-axis data. Note again how the inner and outer solar system points can all be clearly seen. Here the actual logarithms of the variables are plotted. The formula shown in the figure is a best-fit of the data to a straight line. For the uninitiated, het relevant Log-Log details are shown in the boxes.

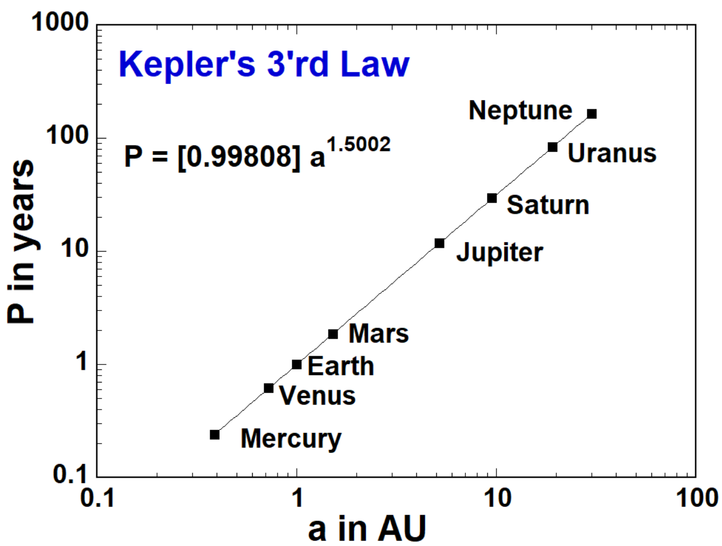


Figure 2a) The Log-Log plot (using logarithmic axis scales) of the period of a planets orbit **P** (in years) and the distance of the planets orbit around the Sun **a** (in AU). The solid line connecting the data represents a best-fit to a power law dependence.

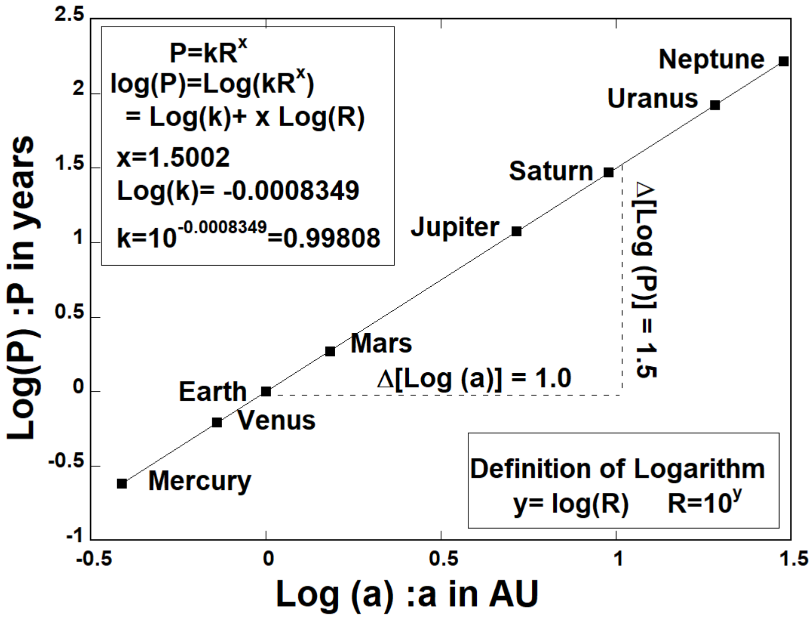


Figure 2b) The Log-Log plot of the period of a planet's orbit **P** (in years) and the distance of the planets orbit around the Sun **a** (in AU). Note here the actual logarithms are plotted so that the linear slope is clear; The solid line connecting the data represents a best-fit to a power law dependence. Note also that the definitions of the variables, power law dependence, and logarithm function are included in boxes in the figure.

Newton's generalization of Kepler's 3rd Law: mass of the central object

Newton was able to drive Kepler's 3rd Law from his three laws of mechanics, and his Universal Gravitational force law. An illustration of this derivation for circular orbits is given in Appendix II. With the valid approximation that the Sun is much much more massive than the planets orbiting Newton's derivation of Kepler's law leads to:

$$\mathbf{P}^2 = \frac{1}{\mathrm{M}}\mathbf{a}^3 \qquad (3)$$

Here M=the mass of the Sun as measured in **multiples of the mass of the Sun**. Recalling Kepler's relations

$$\mathbf{P}^2 = \mathbf{k}^2\,\mathbf{a}^3 \quad (1a) \qquad \text{or} \qquad \mathbf{P} = \mathbf{k}\,\mathbf{a}^{3/2} \quad (1b)$$

one finds

$$\mathbf{k}^2 = \frac{1}{\mathbf{M}} \qquad (4)$$

Of course, for the planets in orbit about the Sun M= 1 and k=1. **But** Newton's derivation should work for all objects in orbit around a central much much more massive object of mass M! In other words, if you observe satellites/moons orbiting around an object, you can use the orbital size and orbital period to determine the mass of the central object.

Application of Kepler's 3rd Law to Jupiter and Saturn moon systems: masses of Jupiter and Saturn

With this mind, and consider the moons of Saturn and Jupiter. Figure 3 shows a Log-Log plot of the orbital-period/semimajor-axis data for the moons in orbit around Saturn and Jupiter. The first point to note is that the slope of the lines in the figure are again accurately 3/2. This means that Newton's universal laws of gravitation and mechanics are indeed universal. What works for the planets orbiting the Sun, works for moons orbiting the planets. The second point to note is that the linear fitted data for Saturn and Jupiter are offset vertically from each other. This means they have different k values [in formulas (1) - (4)]. From formula (4) one now sees that this offset is because Jupiter and Saturn have different masses from each other and from the mass of the Sun. Importantly, these differing k-values can be used to quantitatively estimate the masses of Jupiter and Saturn in multiples of the mass of the Sun. These masses are summarized in Table I below.

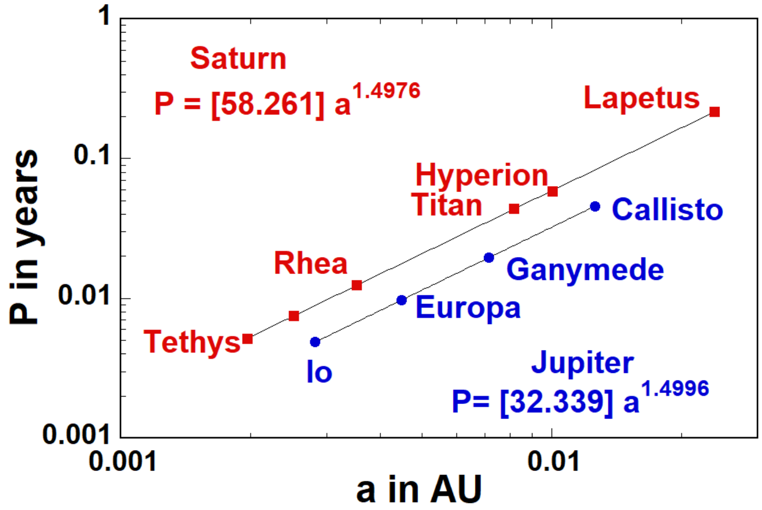


Figure 3) A Log-Log "Kepler type" plot for the moons of Jupiter and Saturn. Here the rather unusual units are used. Specifically, the period of a moons orbit **P** (in years) and the distance of the moon's orbit around the planet **a** (in AU) are used. These units are chosen to make direct contact with the sun-planetary plot in Figure 3. The solid line connecting the data represents a best-fit to a power law dependence.

Application of Kepler's 3rd Law to Earth Moon/satellites system: masses of Earth

The Earth, of course, has the Moon and a plethora of man-made satellites in orbit around it. Kepler's 3rd law should be applicable to the Earth and its Moon/satellites. Figure 4 shows a Kepler Log-Log plot for the Moon and selected satellites in orbit around the earth.

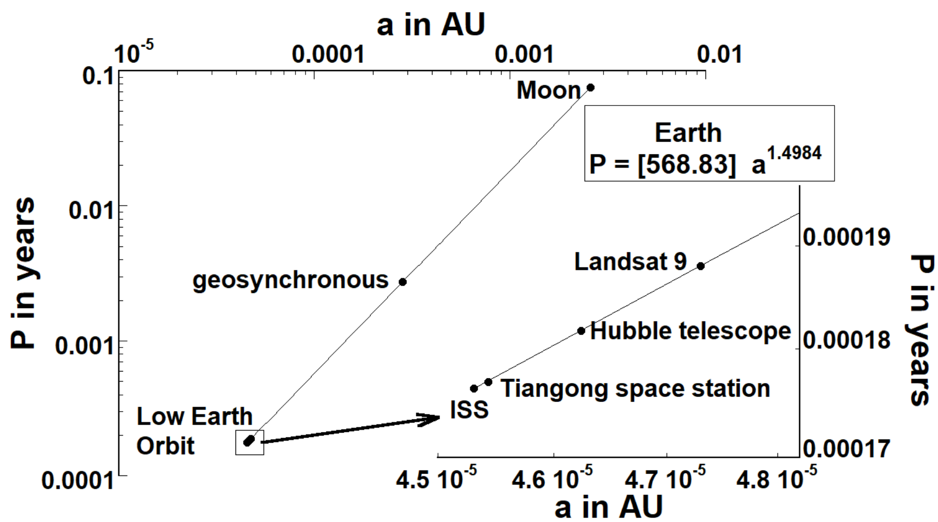


Figure 4) Upper left; A Log-Log "Kepler type" plot for the Earth's Moons and various satellites.
Lower right; An expanded view of the Log-Log "Kepler type" plot for low Earth orbit satellites.
Again, the rather unusual units for the periods **P** (in years) and the distance of the orbit around the Earth **a** (in AU) are used. These units are chosen to make direct contact with the sun-planetary plot in Figure 3. The solid line connecting the data represents a best-fit to a power law dependence.

Summary of the relative masses of the Sun, Earth, Jupiter, and Saturn

Table I: The Kepler-plot fitted k-values, and thereby determined mass ratios for the systems discussed herein. Here M_S and ME are the masses of the Sun and Earth respectively. The M is the mass of the central object in the System (also underlined). Note that disparities from the textbook values are presumably due to the literature values for orbital parameters used. [See Appendix for system data.]

System	k (fitted)	M/M_S	M/M_E	M/M_E (Text: Chaisson)
Sun-planets (Fig. 3)	0.99808	1.00385109	324,813.7	330,000
Jupiter-moons (Fig. 4)	32.339	0.00029461	95.3	95
Saturn-moons (Fig. 4)	58.261	0.0009562	309.4	317
Earth-Moon-satellites (Fig. 3)	568.83	3.0905 $(10)^{-6}$	1	1

Appendix I : How might Kepler have determined his Third Law

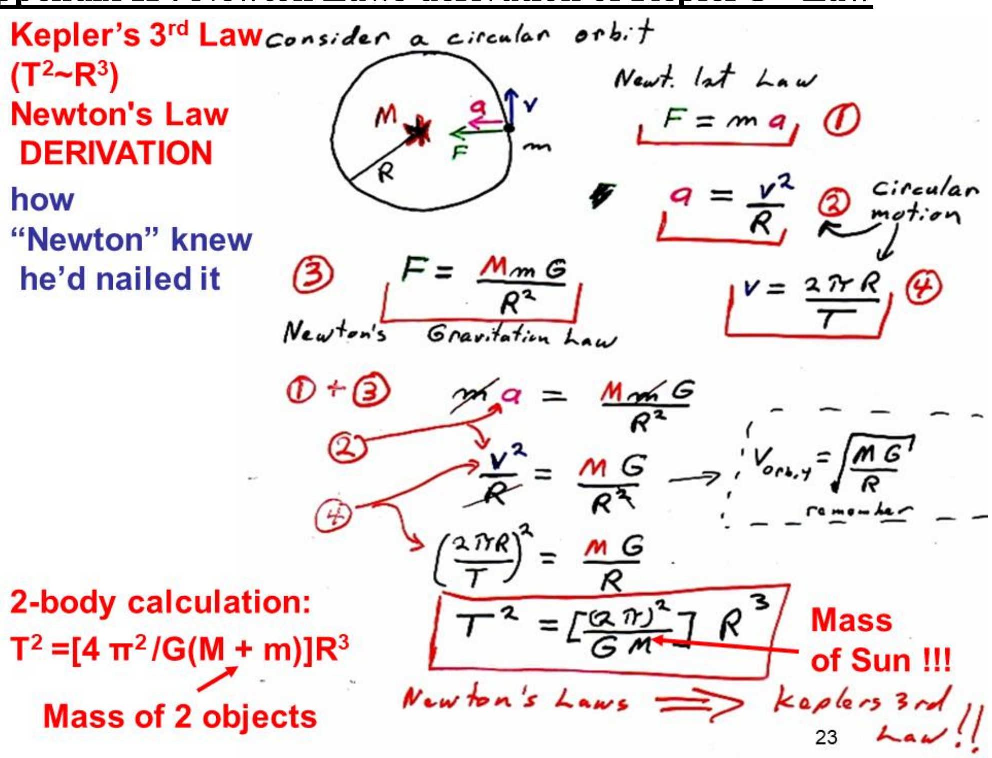
The following is quoted/paraphrased from the online reference
https://astronomy.stackexchange.com/questions/8849/how-did-kepler-guess-his-third-law-from-data
…
Kepler's account of how the third law came to be is as follows (Caspar p.286):
Kepler
"On the 8th of March of this year 1618, if exact information about the time is desired, it appeared in my head. But I was unlucky when I inserted it into the calculation, and rejected it as false. Finally, on May 15, it came again and with a new onset conquered the darkness of my mind, whereat there followed such an excellent agreement between my seventeen years of work at the Tychonic observations and my present deliberation that I at first believed that I had dreamed and assume the sought for in the supporting proofs. But it is entirely certain and exact that **the proportion between the periodic times of any two planets is precisely one and a half times the proportion of the mean distances**."

Although Kepler does not actually describe inspiration that led him to believe this, the curious phrasing provides a very strong clue when combined with some background biographical information:

1. John Napier published *Mirifici Logarithmorum Canonis Descripto* in 1614, which contained the then-new invention of logarithms. Kepler was aware of Napier's work by 1617 (Caspar p. 308), perhaps earlier.
2. Joost Bürgi published work on logarithms almost at the same time as Napier, and Kepler was similarly aware of Bürgi, even praising his mathematical abilities as surpassing most professors of mathematics.

Thus, Kepler's statement is equivalent to saying that the data makes a **slope of 1.5 on a log-log graph**, a which is a very simple linear relationship on this scale.

Appendix II : Newton Laws derivation of Kepler 3rd Law



To be noted.
-A circular orbit of a planet around a much much more massive star is assumed.
-Elliptical orbits have a similar relation.
-This simple derivation assumes the central object is so massive that it can be represented by a central potential energy/force. (1-body problem).
-The 2-body problem generalization, with a the masses of the central and second object is noted.
The 3-body problem (e.g. just one additional planet) can not be solved exactly, so planet-planet interactions are neglected.

One could write $P^2 = a^3 = a^{1+m}$, where the Gravitational Force $F \sim 1/R^m$. Thus, Kepler's value of m=2 Law is a solar system wide verification of the $1/R^2$ Force Law.

Appendix III: System data used

Jupiter System

Moon	Radius of orbit (AU)	Time period of orbit (Years)	Time period In days:	Distance from Jupiter (Km)
Io	2.8182E-03	4.8493E-03	1.77E+00	4.2160E+05
Europa	4.4846E-03	9.7260E-03	3.55E+00	6.7090E+05
Ganymede	7.1524E-03	1.9616E-02	7.16E+00	1.0700E+06
Callisto	1.2587E-02	4.5726E-02	1.67E+01	1.8830E+06

Orbital data on the moons of Saturn:

Moon	Orbital radius (Km)	Orbital period (days)	Orbital radius (AU)	Orbital period (years)	
Mimas	185540	0.942	1.2402E-03	2.5808E-03	
Enceladus	238040	1.37	1.5912E-03	3.7534E-03	
Tethys	294670	1.888	1.9697E-03	5.1726E-03	
Dione	377420	2.737	2.5229E-03	7.4986E-03	
Rhea	527070	4.518	3.5232E-03	1.2378E-02	
Titan	1221870	15.95	8.1676E-03	4.3699E-02	
Hyperion	1500880	21.28	1.0033E-02	5.8301E-02	
Lapetus	3560840	79.33	2.3802E-02	2.1734E-01	

Moon and Satellites

	Distance to earth surface (km)	Orbital radius (km)	Orbital period (mins)	Orbital radius (AU)	Orbital Period (Years)
ISS	408	6779	92.68	4.531E-05	1.763E-04
Tiangong space station	425	6796	93	4.543E-05	1.769E-04
Hubble space telescope	547	6918	95.5	4.624E-05	1.817E-04
Landsat 9	705	7076	98.8	4.730E-05	1.880E-04
Moon		384748	39343.115	2.572E-03	7.485E-02