On the Many-Body Problem of Quantum Theory

English translation of "Zum Mehrkörperproblem der Quantentheorie"

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Abstract

Following Dirac, the symmetric solution of the wave equation in configuration space for a number of electrically interacting particles is represented by a "quantization" of the de Broglie waves in three-dimensional space.

Translator's note. This is a close English translation of the 1927 paper, with the original notation and equation numbering retained as far as possible. The dagger † is used for the authors' "adjoint" q-number notation. Obvious typographical conventions have been rendered in modern LaTeX, without modernizing the physical argument.

The so-called wave equation in configuration space (together with the corresponding matrix-mechanical equations of motion) is, as is well known, a reformulation of the many-body problem of classical point mechanics which satisfies all the requirements of quantum theory and has led to a quantitative treatment, in agreement with experiment, of the most important problems of atomic structure. It lies, however, in the nature of the matter that the range of application of this method is limited, since the classical equations of motion of a number of electrically charged particles can be put into Hamiltonian form only when the finite propagation time of the electromagnetic fields can be neglected. It has therefore not yet been possible to find a treatment of the quantum-mechanical many-body problem appropriate to the nature of relativity theory. In classical theory the difficulty in question is overcome, as is well known, by introducing, in addition to the particle coordinates, the so-called field quantities. These obey special equations, the field equations, which together with the equations of motion satisfy a generalized Hamiltonian principle. By analogy, one may presumably expect a rational solution of the general many-body problem of quantum theory only in connection with a revision of the whole field theory in the spirit of the quantum postulates.

Fundamental to efforts in this direction was Einstein's theory of light quanta, through which the "corpuscle–wave" dualism, especially important in this connection, was first introduced into physics. For the many-body problem, Einstein's work on gas degeneracy is relevant here; following de Broglie, he compared an ideal gas with a system of quantized waves in three-dimensional space.[1] This point of view should therefore be suitable for an attack on the relativistic many-body problem, because matter and field are described mathematically in the same way, namely by partial differential equations. Indeed, in the wave mechanics of the one-body problem, in its relation to field theory, we possess a "classical model" satisfying the requirements of relativity theory. It is therefore natural to ask whether this can be used as a basis for the many-body problem by subjecting the quantities occurring in it—electromagnetic potentials and the Schrödinger wavefunction—to a "quantization."

[1] Cf. the investigations of Born, Heisenberg and Jordan on the fluctuation properties of radiation; also the important contributions in this direction made by Dirac in his investigations of the interaction of radiation and matter, as well as a forthcoming paper by Pauli and Jordan on invariant quantum electrodynamics.

As preparation for such an attempt, we show in the present paper that, from this point of view, in the special case in which the finite propagation velocity of the fields is neglected, one can obtain results agreeing with the wave equation in configuration space. We follow here a paper of Dirac in which the corresponding problem for a number of energetically independent particles was solved.[2] Like Dirac, we restrict ourselves to the case in which the eigenfunctions in configuration space are symmetric in the particles, although, in view of the recent generalization of Dirac's considerations given by one of the authors,[3] the results should probably also remain valid in the antisymmetric case.

Certain fundamental difficulties are, moreover, evidently connected with the problem of a unified treatment of light waves and matter waves. Schrödinger[4] has emphasized these difficulties particularly strongly. They appear in the fact that, with the means used up to now—without a "quantization" of the waves—it is not possible to set up a "closed" system of field equations containing Schrödinger's theory of the hydrogen atom. In particular, some facts to which we shall come in §3 seem to indicate that a natural solution of the difficulties discussed by Schrödinger may also be found by continuing the theory to which we contribute here.

1 Quantization of the de Broglie waves in the case of electrical interaction

We consider a system of electron waves influenced partly by an external electrostatic field and partly by the self-field calculated according to Schrödinger's density hypothesis. We shall neglect both the magnetic interactions and the influence of the finite propagation velocity of the electric field, as well as the relativistic correction. Later we shall substantially generalize these assumptions and, in particular, shall also be able to take magnetic forces into account. To avoid convergence difficulties, we further assume that the waves are contained in a finite closed vessel, at whose walls the wave amplitudes and the electric potential vanish.

We denote a point in space by the radius vector $\mathbf{r}$ drawn to it from an arbitrary fixed point. A function φ of the point in space is accordingly denoted by $\varphi(\mathbf{r})$. Let $\varphi(\mathbf{r},t)$ and $\varphi^*(\mathbf{r},t)$ be the complex-conjugate wave amplitudes, regarded as functions of $\mathbf{r}$ and time t, and let $V(\mathbf{r},t)$ denote the electric potential. Apart from isolated points at which the potential of the external field is singular (point charges), the functions φ and φ^* are then to satisfy

$$\begin{aligned} -\frac{h^2}{8\pi^2\mu}\Delta\varphi - \frac{h}{2\pi i}\frac{\partial\varphi}{\partial t} - eV\varphi = 0,\\ -\frac{h^2}{8\pi^2\mu}\Delta\varphi^* + \frac{h}{2\pi i}\frac{\partial\varphi^*}{\partial t} - eV\varphi^* = 0,\\ \Delta V = 4\pi e\,\varphi^*\varphi. \end{aligned} \tag{1}$$

Here h is Planck's constant, μ the mass, and $-e$ the charge of the electron, while Δ denotes the Laplace differential operator.[5] From the third equation we may put

$$V(\mathbf{r}) = V_0(\mathbf{r}) - e\int G(\mathbf{r},\mathbf{r}')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}')\,dv', \tag{2}$$

where $V_0(\mathbf{r})$ denotes the potential of the "external" field at $\mathbf{r}$, while $G(\mathbf{r},\mathbf{r}')$ is the Green function of Laplace's equation for the region defined by the vessel, and dv' is a volume element about $\mathbf{r}'$.

[2] P. A. M. Dirac, Proc. Roy. Soc. London (A) **114**, 243 (1927).

[3] P. Jordan, Z. Phys. **44**, 473 (1927). (The arguments of that paper are incorrect in two points; a correction will appear shortly.)

[4] E. Schrödinger, Ann. Phys. **82**, 265 (1927).

[5] We restrict ourselves here to a single kind of particle. There is, however, no difficulty in treating several kinds of particles, for example protons and electrons, in an analogous way.

The integration extends over the whole region. If the vessel is large, then to a high approximation

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{3}$$

The system (1) can, as is well known, be derived from a Hamiltonian principle, which we write in the form

$$\delta \int_{t_0}^{t_1} L\,dt = 0, \tag{4}$$

where

$$L = \int dv \left\{ -\frac{h^2}{8\pi^2\mu}(\operatorname{grad}\varphi^* \cdot \operatorname{grad}\varphi) + \frac{h}{4\pi i}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^*}{\partial t}\varphi\right) + eV\varphi^*\varphi - \frac{1}{8\pi}[\operatorname{grad}(V-V_0)]^2 \right\}, \tag{5}$$

and the variations of φ^*, φ, V are to vanish on the boundary of the fixed space-time integration region.

Let $u_s(\mathbf{r})$ be the normalized eigenfunctions of

$$\frac{h^2}{8\pi^2\mu}\Delta u + (E + eV_0)u = 0, \tag{6}$$

and let E_s be the corresponding eigenvalues, i.e. the energy values when only the external field is present. We expand

$$\begin{aligned} \varphi(\mathbf{r},t) &= \sum_s b_s(t)u_s(\mathbf{r}), \\ \varphi^*(\mathbf{r},t) &= \sum_s b_s^*(t)u_s^*(\mathbf{r}). \end{aligned} \tag{7}$$

Here b_s and b_s^* are mutually complex-conjugate functions of time alone. Substitution in L, using (2), (6), and normalization, gives

$$M = \int L\,dv = -\sum_s E_s b_s^* b_s + \frac{e^2}{2}\sum_{\mu\nu rs} A(\mu\nu|rs) b_\mu^* b_\nu^* b_r b_s + \frac{h}{4\pi i}\sum_s \left(b_s^*\dot{b}_s - \dot{b}_s^* b_s\right), \tag{8}$$

where

$$A(\mu\nu|rs) = \iint G(\mathbf{r}',\mathbf{r}'')u_\mu^*(\mathbf{r}')u_\nu^*(\mathbf{r}'')u_r(\mathbf{r}')u_s(\mathbf{r}'')\,dv'\,dv''. \tag{9}$$

The variational principle (4) now takes the form

$$\delta \int M\,dt = 0, \tag{10}$$

which is the form of Hamilton's principle in mechanics.

We next calculate the total energy. By familiar theorems of wave mechanics we may write

$$E = \int dv \left\{ \frac{h^2}{8\pi^2\mu}(\operatorname{grad}\varphi^* \cdot \operatorname{grad}\varphi) - eV_0\varphi^*\varphi + \frac{1}{8\pi}[\operatorname{grad}(V-V_0)]^2 \right\}, \tag{11}$$

whence

$$E = \sum_s b_s^* b_s E_s + \frac{e^2}{2}\sum_{\mu\nu rs} A(\mu\nu|rs) b_\mu^* b_\nu^* b_r b_s. \tag{12}$$

Comparison with (8) gives from (10)

$$-\frac{h}{2\pi i}\frac{db_s}{dt} = \frac{\partial E}{\partial b_s^*}, \qquad \frac{h}{2\pi i}\frac{db_s^*}{dt} = \frac{\partial E}{\partial b_s}. \tag{13}$$

Thus $\frac{h}{2\pi i}b_s$ and b_s^* form a pair of canonical variables.

Following Dirac, we may therefore replace $\frac{h}{2\pi i}b_s$ and b_s^* by conjugate q-numbers $\frac{h}{2\pi i}b_s$ and $b_s^\dagger$. The quantities b_s and $b_s^\dagger$ are not to commute, but satisfy

$$b_\nu b_\mu^\dagger - b_\mu^\dagger b_\nu = \delta_{\mu\nu}. \tag{14}$$

Equations (13) thereby become the q-number equations

$$\frac{h}{2\pi i}\frac{db_s}{dt} = Eb_s - b_sE, \qquad \frac{h}{2\pi i}\frac{db_s^\dagger}{dt} = Eb_s^\dagger - b_s^\dagger E. \tag{15}$$

To make the problem definite, a convention for the order of the factors in E is still required. We return to this question below and, for the moment, use the ordering chosen in (12).

The relations (14) and (15) satisfy the requirements of quantum theory. As we shall now see, these assumptions lead to the solution of the corresponding configuration-space wave equation that is symmetric in the particles. As one of the authors has shown, however, (14) is not the only possible solution of the problem of introducing quantum theory here; there is another possibility which—at least when the interaction of the particles is neglected—leads to the antisymmetric case.

Following Dirac, instead of the complex amplitudes $b_s, b_s^\dagger$ we introduce the real quantities N_s, Θ_s by

$$b_s = e^{-2\pi i\Theta_s/h}N_s^{1/2}, \qquad b_s^\dagger = N_s^{1/2}e^{2\pi i\Theta_s/h}. \tag{16}$$

The eigenvalues N_s' of N_s have the intuitive meaning of the possible numbers of electrons in state s when the mutual coupling of the electrons is removed.[6] We then obtain

$$E = \sum_s N_sE_s \tag{1}$$

$$+ \frac{e^2}{2}\sum_{\mu\nu rs} A(\mu\nu|rs)N_\mu^{1/2}e^{2\pi i\Theta_\mu/h}N_\nu^{1/2}e^{2\pi i\Theta_\nu/h}e^{-2\pi i\Theta_r/h}N_r^{1/2}e^{-2\pi i\Theta_s/h}N_s^{1/2}. \tag{17}$$

The noncommutativity relation becomes

$$N_\mu e^{-2\pi i\Theta_\nu/h} - e^{-2\pi i\Theta_\nu/h}N_\mu = \delta_{\mu\nu}e^{-2\pi i\Theta_\nu/h}. \tag{18}$$

Thus N_ν and Θ_ν are canonically conjugate, and $e^{-2\pi i\Theta_\nu/h}$ may be regarded as an operator which, acting on a function of the N's, replaces N_ν' by $N_\nu' - 1$.

Let $\psi(N_1', N_2', \ldots)$ be the function which, according to the general theory of Dirac and one of the authors, has the meaning that $|\psi|^2$ is the probability function for the quantities N_r' when the mutual coupling of the electrons is removed, and which corresponds to the Hamilton–Jacobi function of classical mechanics. It satisfies the Schrödinger-type equation

$$-\frac{h}{2\pi i}\frac{\partial\psi(N_1', N_2', \ldots)}{\partial t} = E\psi(N_1', N_2', \ldots), \tag{19}$$

where E is given by (17). From the operator property just described one has, schematically,

$$N_\mu'^{1/2}e^{2\pi i\Theta_\mu/h}N_\nu'^{1/2}e^{2\pi i\Theta_\nu/h}N_r'^{1/2}e^{-2\pi i\Theta_r/h}N_s'^{1/2}e^{-2\pi i\Theta_s/h}$$
$$= N_s'^{1/2}(N_r' - \delta_{rs})^{1/2}(N_\nu' + 1 - \delta_{\nu r} - \delta_{\nu s})^{1/2}(N_\mu' + 1 + \delta_{\mu\nu} - \delta_{\mu r} - \delta_{\mu s})^{1/2}e^{\frac{2\pi i}{h}(\Theta_\mu + \Theta_\nu - \Theta_r - \Theta_s)}. \tag{20}$$

[6] $b_s^\dagger$ is obtained from b_s not merely by replacing i by $-i$, but also by reversing the order of multiplication. We therefore use the dagger notation for such "adjoint" quantities.

Consequently

$$-\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = \sum_s N'_s E_s \psi \tag{2}$$

$$+\frac{e^2}{2}\sum_{\mu\nu rs} A(\mu\nu|rs) N_s'^{1/2}(N'_r-\delta_{rs})^{1/2}(N'_\nu+1-\delta_{\nu r}-\delta_{\nu s})^{1/2}$$

$$\times (N'_\mu+1+\delta_{\mu\nu}-\delta_{\mu r}-\delta_{\mu s})^{1/2} e^{\frac{2\pi i}{h}(\Theta_\mu+\Theta_\nu-\Theta_r-\Theta_s)}\psi. \tag{21}$$

Equation (21) shows that the eigenvalues of the N_r are indeed integers. It also follows readily from (15) that $N=\sum_s N_s = \sum_s b_s^\dagger b_s$ is constant in time, i.e. forms a diagonal matrix, corresponding to conservation of electricity in the "classical" equations (1). If there is only a single particle, say

$$N'_r = \delta_{ro}, \tag{22}$$

the operator representing the interaction energy must reduce to zero. Indeed, only terms with $s=o$ can remain, and these contain a factor which vanishes by (22).

2 Comparison with the wave equation in configuration space

To justify (21) for several particles we compare it with Dirac's representation, based on Schrödinger's method, of the many-body problem with symmetric eigenfunctions. Number the particles by $k=1,2,\ldots,N$. Schrödinger's equation is

$$\sum_k \left\{ -\frac{h^2}{8\pi^2\mu}\Delta_k - eV_0(\mathbf{r}_k) + \frac{e^2}{2}\sum_{l\neq k} G(\mathbf{r}_k,\mathbf{r}_l)\right\}\Psi - \frac{h}{2\pi i}\frac{\partial\Psi}{\partial t} = 0. \tag{23}$$

Here Δ_k is the Laplacian belonging to the coordinates of the kth particle, $\mathbf{r}_k$ its coordinate vector, and $G(\mathbf{r}_k,\mathbf{r}_l)$ the Green function for the positions of particles k and l.

We solve (23) with

$$\Psi = \sum b(n_1,n_2,\ldots)u_{n_1}(\mathbf{r}_1)u_{n_2}(\mathbf{r}_2)\cdots, \tag{24}$$

where the u_n are the eigenfunctions of the unperturbed one-electron problem as in the preceding section. The coefficient $b(n_1,n_2,\ldots)$ must be symmetric in its arguments. Substitution in (23), using (6), gives

$$-\frac{h}{2\pi i}\frac{\partial b(n_1,n_2,\ldots)}{\partial t} = \sum_k E_{n_k} b(n_1,n_2,\ldots) \tag{3}$$

$$+\frac{e^2}{2}\sum_{k\neq l}\sum_{m_k m_l} A(m_k m_l|n_k n_l) b(n_1,\ldots,m_k,\ldots,m_l,\ldots). \tag{25}$$

Because b is symmetric in $n_1,n_2,\ldots$, it can instead be regarded as a function of the numbers $N'_1,N'_2,\ldots$, where N'_r is the number of the n_k equal to r. Thus N'_r has the same meaning as in the preceding section. Since $|b|^2$ is a state probability, the normalization must be changed; following Dirac, put

$$b(N'_1,N'_2,\ldots) = \left(\frac{N'!}{N'_1!N'_2!\cdots}\right)^{1/2} b(n_1,n_2,\ldots), \qquad N' = \sum_r N'_r. \tag{26}$$

Then, also when indices coincide,

$$b(n_1,\ldots,n_{k-1},m_k,n_{k+1},\ldots,n_{l-1},m_l,n_{l+1},\ldots)$$

$$= e^{\frac{2\pi i}{h}(\Theta_{n_k}+\Theta_{n_l}-\Theta_{m_k}-\Theta_{m_l})}\left(\frac{N'_1!N'_2!\cdots}{N'!}\right)^{1/2} b(N'_1,N'_2,\ldots). \tag{27}$$

Here, as before, $e^{2\pi i\Theta_r/h}$ denotes the shift operator associated with N'_r.

Thus

$$-\frac{h}{2\pi i}\frac{\partial b(N'_1,N'_2,\ldots)}{\partial t} = b(N'_1,N'_2,\ldots)\sum_r N'_r E_r + \frac{e^2}{2}\sum_{k\neq l}\sum_{m_k m_l} A(m_k m_l|n_k n_l)(N'_1!N'_2!\cdots)^{1/2} \times e^{\frac{2\pi i}{h}(\Theta_{n_k}+\Theta_{n_l}-\Theta_{m_k}-\Theta_{m_l})}(N'_1!N'_2!\cdots)^{-1/2}b(N'_1,N'_2,\ldots). \quad (28)$$

We perform the summation by first, for fixed k,l, letting $\mu = m_k$ and $\nu = m_l$ run over all values. We then collect the terms for which $n_k = r$ and $n_l = s$. For a given set $n_1, n_2, \ldots$ there are $N'_r N'_s$ such terms when $r \neq s$, and $N'_r(N'_r-1)$ when $r = s$; in the former case a factor $1/2$ must be included to avoid counting every electron pair twice. Hence

$$-\frac{h}{2\pi i}\frac{\partial b}{\partial t} = \sum_r N'_r E_r b + \frac{e^2}{2}\sum_{\mu\nu rs} A(\mu\nu|rs)N'_r(N'_s-\delta_{rs})(N'_1!N'_2!\cdots)^{-1/2} \times e^{\frac{2\pi i}{h}(\Theta_r+\Theta_s-\Theta_\mu-\Theta_\nu)}(N'_1!N'_2!\cdots)^{1/2}b. \quad (29)$$

But the factorial and shift-operator factor is equal to the square-root factor occurring in (21):

$$N'_r(N'_s-\delta_{rs})(N'_1!N'_2!\cdots)^{-1/2}e^{\frac{2\pi i}{h}(\Theta_r+\Theta_s-\Theta_\mu-\Theta_\nu)}(N'_1!N'_2!\cdots)^{1/2} = \{N'_s(N'_r-\delta_{rs})\}^{1/2}\{(N'_\nu+1-\delta_{\nu r}-\delta_{\nu s})(N'_\mu+1+\delta_{\mu\nu}-\delta_{\mu r}-\delta_{\mu s})\}^{1/2}e^{\frac{2\pi i}{h}(\Theta_r+\Theta_s-\Theta_\mu-\Theta_\nu)}. \quad (30)$$

Thus (29) is the same equation as (21), or

$$\psi(N'_1,N'_2,\ldots) = b(N'_1,N'_2,\ldots). \quad (31)$$

3 On the order of the factors in the energy expression

We now make some remarks on the order of the factors in (12), returning to the variational principle (4). Let $\eta(\mathbf{r}'-\mathbf{r}'') = \eta(\mathbf{r}''-\mathbf{r}')$ be the singular matrix defined by

$$\int \eta(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r}')\,dv' = \operatorname{grad}\varphi(\mathbf{r}). \quad (32)$$

We may then write

$$\int(\operatorname{grad}\varphi^*\cdot\operatorname{grad}\varphi)\,dv = -\iint \eta(\mathbf{r}'-\mathbf{r}'')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}'')\,dv'\,dv'', \quad (33)$$

and consequently

$$L = \int dv\,\frac{h}{4\pi i}\left\{\varphi^*(\mathbf{r})\frac{\partial\varphi(\mathbf{r})}{\partial t} - \frac{\partial\varphi^*(\mathbf{r})}{\partial t}\varphi(\mathbf{r})\right\} - \frac{h^2}{8\pi^2\mu}\iint \eta(\mathbf{r}'-\mathbf{r}'')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}'')\,dv'\,dv'' + e\int V_0\varphi^*\varphi\,dv + \frac{e^2}{2}\iint G(\mathbf{r}',\mathbf{r}'')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}')\varphi^*(\mathbf{r}'')\varphi(\mathbf{r}'')\,dv'\,dv''. \quad (34)$$

Similarly,

$$E' = -\frac{h^2}{8\pi^2\mu}\iint \eta(\mathbf{r}'-\mathbf{r}'')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}'')\,dv'\,dv'' - e\int V_0\varphi^*\varphi\,dv + \frac{e^2}{2}\iint G(\mathbf{r}',\mathbf{r}'')\varphi^*(\mathbf{r}')\varphi(\mathbf{r}')\varphi^*(\mathbf{r}'')\varphi(\mathbf{r}'')\,dv'\,dv''. \tag{35}$$

In this form (4) is Hamilton's principle for a system of infinitely many unknowns, in which $\frac{h}{2\pi i}\varphi(\mathbf{r})$ and $\varphi^*(\mathbf{r})$ for all points $\mathbf{r}$ play the role of canonical variables. Passing to the limit from finitely many unknowns leads to the quantum conditions

$$\varphi(\mathbf{r}')\varphi^\dagger(\mathbf{r}'') - \varphi^\dagger(\mathbf{r}'')\varphi(\mathbf{r}') = \delta(\mathbf{r}'-\mathbf{r}''), \tag{36}$$

where δ is Dirac's symbolic function, defined for an arbitrary function F by

$$F(\mathbf{r}) = \int F(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})\,dv'. \tag{37}$$

Indeed, using (36), the equations for φ and $\varphi^\dagger$, whose classical analogues follow from (4), can be written

$$\frac{h}{2\pi i}\frac{\partial\varphi(\mathbf{r})}{\partial t} = E\varphi(\mathbf{r}) - \varphi(\mathbf{r})E, \qquad \frac{h}{2\pi i}\frac{\partial\varphi^\dagger(\mathbf{r})}{\partial t} = E\varphi^\dagger(\mathbf{r}) - \varphi^\dagger(\mathbf{r})E. \tag{38}$$

Thus the requirements of quantum theory are indeed fulfilled. If (7) is replaced by the corresponding q-number equations

$$\varphi(\mathbf{r}) = \sum_s b_s u_s(\mathbf{r}), \qquad \varphi^\dagger(\mathbf{r}) = \sum_s b_s^\dagger u_s^*(\mathbf{r}), \tag{7'}$$

one can pass directly from (14) to (36), and conversely.[7]

In (35) we chose the order of factors that arises naturally if the system (1) is simply carried over into quantum theory. This ordering is not correct, however, as comparison with (12), using (7), shows. In the last term of (35), when the c-numbers φ, φ^* are replaced by q-numbers $\varphi, \varphi^\dagger$, the two middle factors $\varphi(\mathbf{r}')$ and $\varphi^\dagger(\mathbf{r}'')$ must be interchanged. From (36),

$$\varphi(\mathbf{r}')\varphi^\dagger(\mathbf{r}'') = \varphi^\dagger(\mathbf{r}'')\varphi(\mathbf{r}') + \delta(\mathbf{r}'-\mathbf{r}'').$$

If the energy in (35) is denoted by E' and the correct expression by E, then, momentarily disregarding the singularity of $G(\mathbf{r}',\mathbf{r}'')$ at $\mathbf{r}'=\mathbf{r}''$,

$$E = E' - \frac{e^2}{2}\int G(\mathbf{r},\mathbf{r})\varphi^\dagger(\mathbf{r})\varphi(\mathbf{r})\,dv. \tag{39}$$

This expression has a simple interpretation. Classically, the interaction energy of a system of point masses, when every pair has interaction energy $e^2G(\mathbf{r}_k,\mathbf{r}_l)$, is

$$\frac{e^2}{2}\sum_{k,l}{}' G(\mathbf{r}_k,\mathbf{r}_l) = \frac{e^2}{2}\sum_{k,l} G(\mathbf{r}_k,\mathbf{r}_l) - \frac{e^2}{2}\sum_k G(\mathbf{r}_k,\mathbf{r}_k).$$

If $N(\mathbf{r}')dv' = \varphi^\dagger(\mathbf{r}')\varphi(\mathbf{r}')dv'$ point masses lie in dv', the sum can be replaced by

$$\frac{e^2}{2}\iint G(\mathbf{r}',\mathbf{r}'')N(\mathbf{r}')N(\mathbf{r}'')\,dv'\,dv'' - \frac{e^2}{2}\int G(\mathbf{r},\mathbf{r})N(\mathbf{r})\,dv. \tag{40}$$

[7]This relation between (7'), (14), and (36) will be discussed in greater detail in a subsequent paper by Jordan. For the generalization of these equations in relativistically invariant quantum electrodynamics, see a forthcoming paper by Pauli and Jordan.

The interaction-energy expression used above is therefore exactly what one expects from a meaningful analogy with the classical theory. Remarkably, the noncommutative multiplication of quantum mechanics makes it possible to express the difference between a double and a single volume integral in (40) by a single double volume integral. In this way quantum mechanics permits one to express analytically, in a simple form, that the "self-field" of an electron does not act on that electron in the same way as an "external field"—a distinction that in classical theory appeared very unsatisfactory and difficult to formulate exactly.

4 Treatment of general interactions between the particles

We now proceed to include the magnetic interactions between the electrons insofar as they are of order $1/c^2$. Correspondingly, terms proportional to $1/c^2$ in the energy of each individual particle, arising from the relativistic variability of mass, must also be included. In addition, retardation of the electric (not magnetic) forces already gives energy corrections of order $1/c^2$; we return to these corrections at the end.

Before considering this problem more closely, we note that the results of the preceding sections permit a very extensive generalization. Consider a many-body problem whose energy in corpuscular representation has the form

$$H = \sum_{k=1}^{N} H_1(p_k, q_k) + \frac{1}{2} \sum_{k,l=1}^{N}{}' H_2(p_k, q_k; p_l, q_l). \tag{41}$$

Here the one-particle energy $H_1(p_k, q_k)$—where p_k, q_k denote all momenta and coordinates belonging to the kth particle—may be chosen arbitrarily (but in the same way for all particles). Between every two point masses there is likewise an arbitrarily chosen interaction H_2, with

$$H_2(p_k, q_k; p_l, q_l) = H_2(p_l, q_l; p_k, q_k).$$

The prime on the sum means that terms with $k = l$ are excluded. These assumptions suffice for the arguments developed above. If only symmetric eigenfunctions are considered, (41) may be replaced by an equivalent wave problem with quantized waves. The energy of the whole system can be expressed in terms of $b_r^\dagger, b_s$, or equivalently the quantized wave amplitudes $\varphi^\dagger(\mathbf{r}), \varphi(\mathbf{r})$ connected with them by (7'), as

$$H = \int dv'\, \varphi^\dagger(\mathbf{r}') H_1\left(\frac{h}{2\pi i}\frac{\partial}{\partial q'}, q'\right) \varphi(\mathbf{r}') \tag{6}$$

$$+ \frac{1}{2} \iint dv'\, dv''\, \varphi^\dagger(\mathbf{r}')\varphi^\dagger(\mathbf{r}'') H_2\left(\frac{h}{2\pi i}\frac{\partial}{\partial q'}, q'; \frac{h}{2\pi i}\frac{\partial}{\partial q''}, q''\right) \varphi(\mathbf{r}')\varphi(\mathbf{r}''). \tag{42}$$

The proof is literally the same as for the special case above.

If we now try to determine the magnetic interaction of the electrons, the previous methods do not give a unique prescription. Classically, as Darwin showed,[8] these interactions can be calculated from a Hamiltonian function of the form

$$H_{kl} = -\frac{e^2}{2\mu^2 c^2} \frac{\mathbf{p}_k \cdot \mathbf{p}_l}{r_{kl}}, \tag{43}$$

where $\mathbf{p}_k, \mathbf{p}_l$ are the electron momentum vectors and $r_{kl} = |\mathbf{q}_k - \mathbf{q}_l|$. With the usual methods based on matrix theory or Schrödinger's wave equation in configuration space, an exact quantum-theoretical translation of this equation is difficult because noncommuting quantities are multiplied in H_{kl}; close analogy with the classical formula (43) is insufficient to determine their order.[9]

[8] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

[9] No such difficulty occurs in problems of the type $H = p^2/(2\mu) - e^2/r$.

We shall see, however, that representing the electrons by quantized waves necessarily leads to a definite expression for the magnetic interaction energy. Bringing that expression into the form (42), our general theorem then allows us to infer backwards the form (41) and hence to give the exact quantum-mechanical analogue of (43).

First use the φ waves in their unquantized form (the Schrödinger waves of a single electron). If $\mathbf{j}(\mathbf{r})$ is the electric current density, the magnetic interaction energy, according to the Biot–Savart law (and also from the wave-mechanical variational principle), is

$$H_m = -\frac{1}{2c^2}\iint [\mathbf{j}(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}'')]G(\mathbf{r}',\mathbf{r}'')\,dv'\,dv''. \tag{44}$$

In the approximation relevant here we put

$$\mathbf{j} = \frac{eh}{2\mu}\frac{1}{2\pi i}\left(\varphi^*\operatorname{grad}\varphi - (\operatorname{grad}\varphi^*)\varphi\right). \tag{45}$$

Passing from the c-numbers φ^*, φ to the q-numbers $\varphi^\dagger, \varphi$, (44) becomes

$$\begin{aligned} H_m = -\frac{e^2h^2}{32\pi^2\mu^2c^2}\iint dv'\,dv''\,G(\mathbf{r}',\mathbf{r}'')\{&\varphi^\dagger(\mathbf{r}')\varphi^\dagger(\mathbf{r}'')\operatorname{grad}\varphi(\mathbf{r}')\cdot\operatorname{grad}\varphi(\mathbf{r}'') \\ &- \varphi^\dagger(\mathbf{r}')\operatorname{grad}\varphi^\dagger(\mathbf{r}'')\cdot\operatorname{grad}\varphi(\mathbf{r}')\,\varphi(\mathbf{r}'') \\ &- (\operatorname{grad}\varphi^\dagger(\mathbf{r}'))\varphi^\dagger(\mathbf{r}'')\,\varphi(\mathbf{r}')\cdot\operatorname{grad}\varphi(\mathbf{r}'') \\ &+ (\operatorname{grad}\varphi^\dagger(\mathbf{r}'))\cdot(\operatorname{grad}\varphi^\dagger(\mathbf{r}''))\varphi(\mathbf{r}')\varphi(\mathbf{r}'')\}. \end{aligned} \tag{46}$$

By partial integration this can be put into the form (42):

$$H_m = -\frac{e^2h^2}{32\pi^2\mu^2c^2}\iint dv'\,dv''\,\varphi^\dagger(\mathbf{r}')\varphi^\dagger(\mathbf{r}'')\,\Omega\,\varphi(\mathbf{r}')\varphi(\mathbf{r}''), \tag{47}$$

where

$$\begin{aligned} \Omega &= G\operatorname{grad}'\cdot\operatorname{grad}'' + (\operatorname{grad}' G)\cdot\operatorname{grad}'' + (\operatorname{grad}'' G)\cdot\operatorname{grad}' + G\operatorname{grad}'\cdot\operatorname{grad}'' \\ &= 2\left[G\operatorname{grad}'\cdot\operatorname{grad}'' + (\operatorname{grad}' G)\cdot\operatorname{grad}'' + (\operatorname{grad}'' G)\cdot\operatorname{grad}'\right]. \end{aligned} \tag{48}$$

Here grad' and grad'' mean gradients acting on functions of $\mathbf{r}'$ and $\mathbf{r}''$, respectively. The equality of the two expressions in (48) follows from

$$\operatorname{grad}'\cdot\operatorname{grad}'' G = -\operatorname{div}'\operatorname{grad}' G = 0. \tag{49}$$

From (48) we obtain the quantum-mechanical analogue of (43):

$$H_{kl} = -\frac{e^2}{4\mu^2c^2}\left\{\frac{1}{r_{kl}}\,\mathbf{p}_k\cdot\mathbf{p}_l + \mathbf{p}_k\cdot\mathbf{p}_l\,\frac{1}{r_{kl}}\right\}. \tag{50}$$

Although this formula was derived from considerations assuming Bose–Einstein rather than Pauli statistics for electrons, it should be valid independently of this assumption, since (50) is plainly of a form that no longer refers to any special choice of statistics.

Finally, as regards the effects proportional to $1/c^2$ arising from retardation of the electric interactions, we merely emphasize that, following the cited investigation of Darwin, they can be treated in precisely the same way as the magnetic interactions.

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