INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2003

Questions 7: Final take-home exam. Due 5pm Friday Dec 19th .

Do any two of the following questions. Answers should never involve anything more complex than a single closed integral. If you wish to use Mathematica or Maple to illustrate results, feel free to do so. Please try to explain all derivations, with diagrams and sketches when possible. I will give lots of credit for answers that are conceptually correct! Please do not consult with one-another, but do not hesitate to consult reference sources, or to come to me individually should you become stuck.

1. (a) What does the field operator $\psi_\sigma(\vec{x},t)$ of a particle represent in quantum field theory?
   (b) How is the Pauli Exclusion principle built into the properties of the field operator $\psi_\sigma(\vec{x},t)$?
   (c) Using a second-quantized notation, write down the operators or states for
      i. The charge density of a gas of identical particles.
      ii. The spin density of a gas of identical particles
      iii. The Coulomb interaction in a degenerate Coulomb gas.
      iv. The Kinetic energy density of a gas of identical particles.
      v. The particle current operator.
      vi. The wavefunction of N particles $\psi(\vec{x}_1\sigma_1,\dots\vec{x}_N\ \sigma_N,t)$
   (d) Explain how the Schrodinger many body equation can be derived starting from the Heisenberg equation of motion of the quantum field operator $H,\psi_\sigma(\vec{x},t)$
   (e) What is the general relationship between the particle current operator and the particle density operator, and how would you prove this relationship?

2. In class we discussed how the static charge susceptibility of an electron gas is given by $\chi(\vec{q}) = 2N(0)F(\frac{q}{2k_F})$ where $N(0)$ is the density of states per spin and

$$F(x) = \frac{1}{4x}(1-x^2)\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2} \tag{1}$$

   is the Lindhardt function.

   (a) What is the origin of this relationship, and how does $\chi(q)$ relate the change in the density of electrons to small changes in the electron potential $\delta U(x)$?
   (b) Suppose an impurity is introduced into the electron gas, which gives rise to a "weak" delta function repulsive scattering potential

$$U(\vec{x}) = \lambda\delta^3(\vec{x}) \qquad (\lambda > 0) \tag{2}$$

   Describe and calculate in as much detail as possible, to linear order in λ, the change in the density of electrons surrounding the impurity.
   (c) Can you say anything about the charge of the impurity?

3. The one dimensional electron gas is prone to the development of charge-density wave instabilities. The treatment of these instabilities can be done by the method of mean-field theory. Suppose we have a one-dimensional conductor, described by the Hamiltonian

$$\begin{aligned} H-\mu N &= H_o + H_I, \\ H_o &= -t\sum_{j,\ \sigma}\Big(\psi^\dagger{}_{j+1\ \sigma}\psi_{j\ \sigma} + \psi^\dagger{}_{j\ \sigma}\psi_{j+1\ \sigma}\Big), \\ H_I &= -\frac{g}{2}\sum_j(:\hat{n}_j^2:-1) \end{aligned} \tag{3}$$

where $g>0$ and $\psi^\dagger{}_{j\sigma}$ creates an electron with spin $\sigma = \pm\frac{1}{2}$ at site j. The separation between sites is taken to be unity and the chemical potential has been chosen to be *zero*, giving a half-filled band.

(a) Show that H_o can be diagonalized in the form

$$H_o = \sum_{k\ \sigma}(-2t\cos k)\ c^\dagger{}_{k\sigma}c_{k\sigma}, \tag{4}$$

where $c_{k\sigma} = \frac{1}{\sqrt{N}}\sum_j \psi_{j\sigma}e^{-ikj}$, $k = \frac{2\pi}{N}(0,\ 1,\ \dots N-1)$. Note that the Fermi surfaces are separated by a distance π in momentum space, with average electron density 1 per site.

(b) Suppose a staggered potential $V_j = (-1)^j\Phi$ is applied to the conductor. This will induce a charge density wave

$$\langle n_j\rangle = 1-(-1)^j\Delta_j/g \tag{5}$$

This staggered charge density, will, via the interaction term produce its own staggered potential with the result that even when the external potential is removed, the charge density wave remains. Show that if the fluctuations of the charge density about its average can be ignored, i.e if $\delta n_j = \hat{n}_j - \langle n_j\rangle$ is small, then the the interaction Hamiltonian can be approximated as follows

$$H_I \rightarrow \sum_j\left((-1)^j\Delta_j\hat{n}_j + \frac{\Delta_j^2}{2g}\right) + O(\delta\hat{n}_j^2). \tag{6}$$

(c) Calculate the excitation spectrum in the presence of the uniformly staggered order parameter $\Delta_j = \Delta$. (Hint - you may find it useful to work with the spinor $\Psi_{k\sigma} = \begin{pmatrix} c_{k\sigma} \\ c_{k+\pi\sigma}\end{pmatrix}$. The Hamiltonian becomes a two by two matrix in this basis and is easily diagonalized.)

(d) Now calculate the ground-state energy per site as a function of Δ, $E[\Delta]$, taking into account the term Δ^2/g from (b). Plot your result as a function of Δ and comment on the significance of the result you obtain.