INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Solutions to Problems 7 .

1. (a) Now

$$\alpha^2(\nu)F(\nu) = \frac{1}{\int dS}\int dS \int \frac{dS'}{v'_F}\delta(\nu - \omega_{\mathbf{k}-\mathbf{k}'})|g_{\mathbf{k}-\mathbf{k}'}|^2 \tag{1}$$

For a spherically symmetric Fermi surface, there is no need to average over $\mathbf{k}$, so

$$\alpha^2(\nu)F(\nu) = \int \frac{dS'}{v'_F}\delta(\nu - \omega_{\mathbf{k}-\mathbf{k}'})|g_{\mathbf{k}-\mathbf{k}'}|^2 \tag{2}$$

For an Einstein phonon, the angular integral gives $\frac{dS'}{v'_F} \to N(0)$

$$\alpha^2(\nu) = N(0)\delta(\nu - \omega_0)|g_0|^2 \tag{3}$$

Now since $\lambda = 2\int \alpha^2 F(\nu)/\nu$, we have

$$\alpha^2 F(\nu) = \frac{\lambda\omega_0}{2}\delta(\nu - \omega_0) \qquad \text{(Einstein phonon)}$$

By contrast, for an acoustic phonon,

$$\alpha^2 F(\nu) = \int \frac{dS'}{v_F}\delta(\nu - c|\mathbf{k}-\mathbf{k}'|)g(c|\mathbf{k}-\mathbf{k}'|)^2$$

We may replace $|\mathbf{k}-\mathbf{k}'| = 2k_F\sin(\theta/2)$, and $\int \frac{dS'}{v_F} \to N(0)\int \frac{\sin\theta d\theta}{2}$, so that

$$\begin{aligned}
\alpha^2 F(\nu) &= N(0)\int \frac{\sin\theta d\theta}{2}\delta(\nu - 2ck_F\sin(\theta/2))g(q)^2 \\
&= N(0)\int \frac{\sin\theta d\theta}{2}\frac{1}{c\sin\frac{\theta_0}{2}k_F}\delta(\theta-\theta_0)g\left(\frac{\nu}{c}\right)^2 \\
&= N(0)\frac{g^2[\nu/c]}{ck_F}\left(\frac{\nu}{ck_F}\right)
\end{aligned} \tag{4}$$

Now $g(\nu)^2 \propto \nu$, so that

$$\alpha^2 F(\nu) = \lambda\frac{\nu^2}{\omega_D^2} \qquad \text{(acoustic phonons)}$$

(b) The relaxation rate at absolute zero is given by

$$\begin{aligned}
\Gamma(\omega) = \frac{1}{\tau}(\omega) &= 2Im\Sigma(\omega - i\delta) = 2\pi\int d\nu\alpha^2 F(\nu)(1 - f(\omega-\nu)) \\
&= 2\pi\int_0^\omega d\nu\alpha^2 F(\nu)
\end{aligned} \tag{5}$$

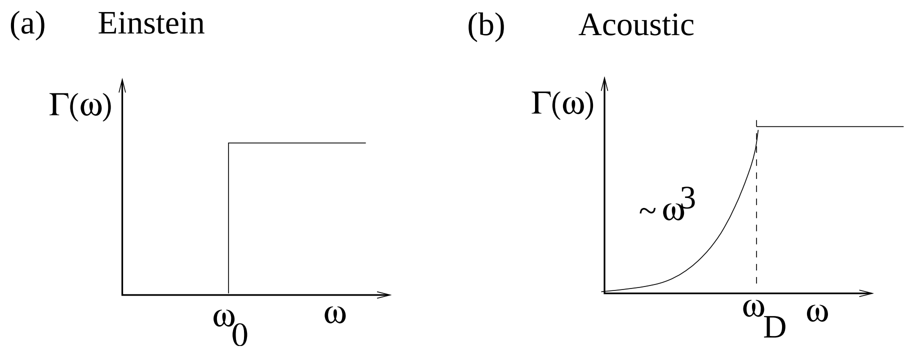

Figure 1: Frequency dependence of the scattering rate for (a) Einstein phonon and (b) acoustic phonon.

For an Einstein phonon,

$$\Gamma(\omega) = \pi\omega_0\lambda\theta(\omega - \omega_0)$$

For an acoustic phonon,

$$\Gamma(\omega) = \frac{2\pi\lambda\omega_0}{3} \times \begin{cases} \left(\frac{\omega}{\omega_D}\right)^3 & (\omega < \omega_D) \\ 1 & (\omega > \omega_D) \end{cases}$$

2. This was a very straightfoward, if laborious question to check you had understood the idea behind the spectral decomposition.

(a) We shall sketch the spectral decomposition in detail for $G_r(\mathbf{k},\omega)$, and then generalize our results quickly to the other two Green functions. Let us first expand the expectation value in the definition of the Greens function as a trace over states,

$$G_R(\mathbf{k},t) = -i\sum_\lambda e^{-\beta(E_\lambda - F)}\langle\lambda|\{\psi_\mathbf{k}(t), \psi^\dagger{}_\mathbf{k}(0)\}|\lambda\rangle\theta(t).$$

There are two terms inside the anticommutator which we expand as follows:

$$G_R(\mathbf{k},t) = -i\sum_\lambda e^{-\beta(E_\lambda - F)}\langle\lambda|\psi_\mathbf{k}(t)\psi^\dagger{}_\mathbf{k}(0)|\lambda\rangle\theta(t) - i\sum_\zeta e^{-\beta(E_\zeta - F)}\langle\zeta|\psi^\dagger{}_\mathbf{k}(0)\psi_\mathbf{k}(t)|\zeta\rangle\theta(t).$$

where we have changed the label of the states from λ to ζ in the second term for later convenience. We now introduce the completeness relation $1 = \sum_\zeta |\zeta\rangle\langle\zeta|$ between the creation and annihilation operator in the first term and $1 = \sum_\lambda |\lambda\rangle\langle\lambda|$ in the second, and use the relationship

$$\langle\lambda|\psi_\mathbf{k}(t)|\zeta\rangle = e^{-i(E_\zeta - E_\lambda)t}\langle\lambda|\psi_\mathbf{k}|\zeta\rangle$$

to obtain

$$
\begin{aligned}
G_R(\mathbf{k},t) &= -i\theta(t)\sum_{\lambda,\zeta} e^{-\beta(E_\lambda-F)}\left(\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle + e^{-(E_\zeta-E_\lambda)\beta}\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\right)e^{-i(E_\zeta-E_\lambda)t} \\
&= -i\theta(t)\sum_{\lambda,\zeta} e^{-\beta(E_\lambda-F)}(1+e^{-(E_\zeta-E_\lambda)\beta})|\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle|^2 e^{-i(E_\zeta-E_\lambda)t}. \qquad (6)
\end{aligned}
$$

Introducing the identity $\int d\omega'\delta(\omega'-(E_\zeta-E_\lambda))=1$ into the summation, and interchanging the order of integration with the order of summation, we obtain

$$
G_R(\mathbf{k},t)=\int\frac{d\omega'}{\pi}A(\mathbf{k},\omega')\times -i\theta(t)e^{-i\omega' t}
$$

where

$$
A(\mathbf{k},\omega')=(1+e^{-\beta\omega'})\sum_{\lambda,\zeta}e^{-\beta(E_\lambda-F)}\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle|^2\pi\delta(\omega'-(E_\zeta-E_\lambda)).
$$

is the spectral function. Finally, Fourier transforming $G_R(\mathbf{k},t)$ we obtain

$$
\begin{aligned}
G_R(\mathbf{k},\omega)=\int dt e^{i\omega t}G_R(\mathbf{k},t) &= \int\frac{d\omega'}{\pi}A(\mathbf{k},\omega')\times -i\int_0^\infty dt e^{i(\omega-\omega'+i\delta)t} \\
&= \int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{1}{\omega-\omega'-i\delta}A(\mathbf{k},\omega) \qquad (7)
\end{aligned}
$$

(Notice the small sign error with respect to the original question).
The same procedure can be carried out for G_A. The spectral decomposition of G_A gives

$$
\begin{aligned}
G_A(\mathbf{k},t) &= i\theta(-t)\sum_{\lambda,\zeta}e^{-\beta(E_\lambda-F)}\left(\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle+\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\right)\times e^{-i(E_\zeta-E_\lambda)t} \\
&= \int\frac{d\omega'}{\pi}A(\mathbf{k},\omega')\times i\theta(-t)e^{-i\omega' t} \qquad (8)
\end{aligned}
$$

and Fourier transforming gives

$$
\begin{aligned}
G_A(\mathbf{k},\omega) &= \int dt e^{i\omega t}G_A(\mathbf{k},t) \\
&= \int\frac{d\omega'}{\pi}A(\mathbf{k},\omega')\times\int_{-\infty}^0 dt e^{i(\omega-\omega'-i\delta)t} \\
&= \int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{1}{\omega-\omega'-i\delta}A(\mathbf{k},\omega) \qquad (9)
\end{aligned}
$$

Lastly, for G_K, the minus sign inside the commutator gives

$$
\begin{aligned}
G_K(\mathbf{k},t) &= -i\langle[\psi_{\mathbf{k}}(t),\psi^\dagger{}_{\mathbf{k}}(0)]\rangle \\
&= -i\sum_{\lambda,\zeta}e^{-\beta(E_\lambda-F)}\left(\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle-\langle\zeta|\psi^\dagger{}_{\mathbf{k}}|\lambda\rangle\langle\lambda|\psi_{\mathbf{k}}|\zeta\rangle\right)\times e^{-i(E_\zeta-E_\lambda)t} \\
&= -i\int\frac{d\omega}{\pi}e^{-i\omega t}A(\mathbf{k},\omega)\left(\frac{1-e^{-\beta\omega}}{1+e^{-\beta\omega}}\right) \\
&= -2i\int\frac{d\omega}{2\pi}e^{-i\omega t}A(\mathbf{k},\omega)\tanh\left(\frac{\beta\omega}{2}\right) \qquad (10)
\end{aligned}
$$

from which we read off

$$
G_K(\mathbf{k},\omega)=-2iA(\mathbf{k},\omega)\tanh\left(\frac{\beta\omega}{2}\right)
$$

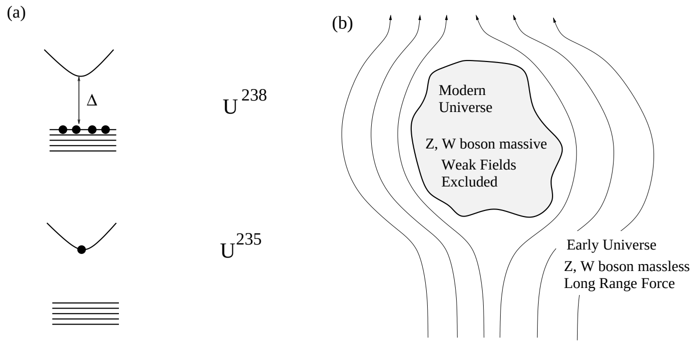

Figure 2: (a) Illustrating why pairing of nucleons leads to an energy gap for adding neutrons to U^{238}, and no energy gap for adding neutrons to U^{235}. (b) The force between electrons and neutrons is weak because the W boson is massive. According to electro-weak theory, this occurs because of a breaking of the $U(1)xSU(2)$ symmetry that exists between the electric and weak force. The weak force becomes short range because of a kind of "cosmic Meissner effect" which makes the Z and W bosons massive, so that the weak force becomes a short-range phenomenon.

(b) From (7) and (9) we obtain

$$
\begin{aligned}
(G_R(\mathbf{k},\omega)-G_A(\mathbf{k},\omega)) &= \int\frac{d\omega'}{\pi}\overbrace{\left[\frac{1}{\omega-\omega'+i\delta}-\frac{1}{\omega-\omega'-i\delta}\right]}^{-i2\pi\delta(\omega-\omega')}A(\mathbf{k},\omega') \\
&= -2iA(\mathbf{k},\omega) \qquad (11)
\end{aligned}
$$

so that in thermal equilibrium the ratio

$$
\left(\frac{G_K(\mathbf{k},\omega)}{G_R(\mathbf{k},\omega)-G_A(\mathbf{k},\omega)}\right)=h=\tanh\left(\frac{\beta\omega}{2}\right)
$$

is independent of momentum. In a non-equilibrium situation, this ratio determines the non-equilibrium distribution function - and in general it will be dependent on momentum.

3. (a) The nuclear forces between nucleons are attractive and short-ranged, and in this sense are very similar to the attraction induced between electrons by virtual phonons inside a superconductor. Like a metal, these attractive forces lead to pairing - in this case nucleon pairing. In U^{238} there are an even number of neutrons, whereas in U^{235} there is an odd-number of neutrons. When one adds a neutron to U^{238}, it is unpaired, costing an energy equal to the pairing or "gap" energy of the paired nucleons, and this is why it requires high energy neutrons to fission U^{238}, rendering it useless as a fisionable fuel. By contrast, when one addes

a neutron to U^{235} it pairs with the remaining unpaired neutron, liberating an amount of energy 2Δ - this energy goes to immediately destabilizes the nucleus and cause fision.

(b) The electron interacts with the proton via the electrical force, involving the exchange of massless, virtual photons. The interaction of an electron with a neutron occurs via the weak force, involving the exchange of a massive vector bosons. The weakness of the nuclear force derives from the mass of the W boson: the energy-time Heisenberg Uncertainty principle means that a heavy virtual boson can only be formed for very brief periods of time, inducing only the smallest interaction vertex between electrons and neutrons. In the early universe, the electro magnetic and weak force were equally strong, mediated by massless bosons. The mass of the W boson developed because of kind of "cosmic Meissner effect" in which it is believed, the early universe developed a condensate of Higg's bosons. Like the magnetic Meissner effect, the electro-weak Meissner effect gives the weak W and Z bosons a mass, excluding the weak fields from our universe and making the weak force a short-range force.

(c) Helium-3 becomes a superfluid at low temperatures because the Helium-3 atoms pair to form a pair condensate that is closely analogous to a superconductor. The interaction between the Helium atoms is repulsive and does not favor s-wave pairing. Instead, the Helium atoms prefer to form a Cooper pair which has a node in the spatial wavefunction, which tends to keep the Helium atoms apart, lowering their energy.

4. In this question, I wanted to check that you had understood how to set up a fermionic path integral.

(a) The first step in setting up the path integral is to write the Trace using coherent states. Let us define

$$|c\rangle = \exp[\hat{c}^\dagger c]|0\rangle, \rangle\bar{c}| = \langle 0|\exp[\bar{c}\hat{c}].$$

Now, for a general operator $\hat{A}$, the matrix element between these coherent states is

$$\begin{aligned}\langle\bar{c}|\hat{A}|c\rangle &= A_{oo}\langle\bar{c}|0\rangle\langle 0|c\rangle + A_{01}\langle\bar{c}|0\rangle\langle 1|c\rangle + A_{10}\langle\bar{c}|1\rangle\langle 0|c\rangle + A_{11}\langle\bar{c}|1\rangle\langle 1|c\rangle \\ &= A_{oo} + A_{01}c + A_{10} + A_{11}\bar{c}c\end{aligned} \tag{12}$$

so that the trace may be written

$$Tr[A] = A_{00} + A_{11} = -\int d\bar{c}dce^{\bar{c}c}\langle\bar{c}|\hat{A}|c\rangle \tag{13}$$

Applying this to the partition function,

$$\begin{aligned}Tr[e^{-\beta H}] &= -\int d\bar{c}_3 dc_0 e^{\bar{c}_3 c_0}\langle\bar{c}_3|e^{-\beta H}|c_0\rangle \\ &= \int d\bar{c}_3 d\bar{c}_3 e^{-\bar{c}_3 c_3}\langle\bar{c}_3|e^{-\beta H}|c_0\rangle,\end{aligned} \tag{14}$$

where we have used the definition, $c_3 = -c_0$. We now use the completeness relation

$$1 = \int d\bar{c}dce^{-\bar{c}c}|c\rangle\langle\bar{c}| \tag{15}$$

to introduce two time-slices into the matrix element $\langle\bar{c}_3|e^{-\beta H}|c_0\rangle$, which we write as

$$\langle\bar{c}_3|e^{-\beta H}|c_0\rangle = \int d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \langle\bar{c}_3|e^{-\Delta\tau H}|c_2\rangle\langle\bar{c}_2|e^{-\Delta\tau H}|c_1\rangle\langle\bar{c}_1|e^{-\Delta\tau H}|c_0\rangle e^{-(\bar{c}_2 c_2 + \bar{c}_1 c_1)} \tag{16}$$

so that

$$Tr[e^{-\beta H}] = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \langle\bar{c}_3|e^{-\Delta\tau H}|c_2\rangle\langle\bar{c}_2|e^{-\Delta\tau H}|c_1\rangle\langle\bar{c}_1|e^{-\Delta\tau H}|c_0\rangle e^{-(\bar{c}_3 c_3 + \bar{c}_2 c_2 + \bar{c}_1 c_1)}.$$

Finally, using the expansion of the matrix element in terms of coherent states,

$$\langle\bar{c}_{j+1}|e^{-\Delta\tau H}|c_j\rangle = e^{\alpha\bar{c}_{j+1}c_j} + O(\Delta\tau^2) \tag{17}$$

where $\alpha = (1 - \Delta\tau\epsilon)$, we obtain

$$\begin{aligned}Z_3 &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 e^{\alpha[\bar{c}_3 c_2 + \bar{c}_2 c_1 + \overbrace{\bar{c}_1 c_0}^{-\bar{c}_1 c_3}] - [\bar{c}_3 c_3 + \bar{c}_2 c_2 + \bar{c}_1 c_1]} \\ &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{-(\bar{c}_3, \bar{c}_2, \bar{c}_1)\begin{bmatrix}1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1\end{bmatrix}\begin{pmatrix}c_3 \\ c_2 \\ c_1\end{pmatrix}\right\}\end{aligned} \tag{18}$$

where we have set $c_0 = -c_3$ in the last step.

(b) Since this integral is Gaussian, the integral is given by the determinant of the matrix:

$$Z_3 = \det\begin{bmatrix}1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1\end{bmatrix} = 1 + \alpha^3 \tag{19}$$

(c) Generalizing this result to N time-slices, we obtain

$$\begin{aligned}Z_N &= \det[\mathcal{M}] \\ \mathcal{M} &= \begin{bmatrix}1 & -\alpha & 0 & \dots & 0 \\ 0 & 1 & -\alpha & \dots & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -\alpha \\ \alpha & \dots & \dots & \dots & 1\end{bmatrix} \\ \det[\mathcal{M}] &= 1 + \alpha^N \qquad \text{(by inspection)}\end{aligned} \tag{20}$$

In the limit $N \to \infty$,

$$\alpha^N = \left(1 - \frac{\beta\epsilon}{N}\right)^N \to e^{-\beta\epsilon} \tag{21}$$

so that

$$Z_N \to 1 + e^{-\beta\epsilon} \tag{22}$$