INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Problems 7. (Substituting as a final) Due Mon, Dec 6th

Choose three of the following four. If time permits, do the fourth for extra credit.

1. (a) Calculate the qualitative frequency dependence of $\alpha^2(\omega)F(\omega)$ for an Einstein phonon (where $\omega_{\mathbf{q}} = \omega_0$ and $g_{\mathbf{q}} = g_0$ are constants) and an acoustic phonon, where $\omega_{\mathbf{q}} = cq$ and $g^2_{\mathbf{q}} \propto q$. You may use a spherical Fermi surface, for which

$$\alpha^2(\omega)F(\omega) = \int_{\mathbf{k}} |g_{\mathbf{k}-\mathbf{k}'}|^2 \delta(\epsilon_{\mathbf{k}'})\delta(\omega - \omega_{\mathbf{k}-\mathbf{k}'}).$$

   (b) Sketch the frequency dependence of the electron-phonon scattering rate for each of these cases.

2. Three fermion Greens functions of particular interest in non-equilibrium physics, the retarded, advanced and Keldysh Greens functions, which in equilibrium have the form

$$\begin{aligned} G_R(\mathbf{k}, t) &= -i\langle\{\psi_{\mathbf{k}}(t), \psi^{\dagger}{}_{\mathbf{k}}(0)\}\rangle\theta(t) \\ G_A(\mathbf{k}, t) &= i\langle\{\psi_{\mathbf{k}}(t), \psi^{\dagger}{}_{\mathbf{k}}(0)\}\rangle\theta(-t) \\ G_K(\mathbf{k}, t) &= -i\langle[\psi_{\mathbf{k}}(t), \psi^{\dagger}{}_{\mathbf{k}}(0)]\rangle \end{aligned} \tag{1}$$

   (a) Carry out a spectral decomposition of these Green functions, and show that in the frequency domain

$$\begin{aligned} G_R(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{1}{\omega' - \omega + i\delta} A(\mathbf{k}, \omega) \\ G_A(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{1}{\omega' - \omega - i\delta} A(\mathbf{k}, \omega) \end{aligned} \tag{2}$$

   giving the form for the spectral function $A(\mathbf{k}, \omega)$.

   (b) Derive the fluctuation-dissipation theorem for Fermions, showing that

$$\frac{G_K(\mathbf{k}, \omega)}{[G_R(\mathbf{k}, \omega) - G_A(\mathbf{k}, \omega)]} = h$$

   is independent of momentum. Your answer should relate this quantity to the Fermi function.

3. The theory superconductivity derived from Landau Ginzburg theory and the microscopic BCS theory of superconductivity led to new insights in many other domains of physics: low temperature, nuclear and particle physics. Explain, in layman's language how the ideas of BCS theory are directly connected with

   (a) The reason that U^{238} requires high energy neutrons to undergo fission, yet U^{235} only requires thermal neutrons.

   (b) The reason why electrons interact strongly with protons, yet weakly with neutrons.

(c) The reason that Helium $He-3$, a fluid of fermions, becomes superfluid at $1mK$.

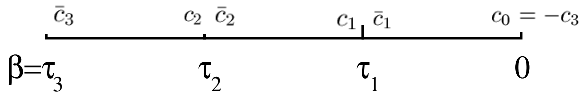
4. (a) Suppose $H = \epsilon c^\dagger c$ represents a single fermion state. Consider the approximation to the partition function obtained by dividing up the period $\tau \in [0,\beta]$ into N equal time-slices,

$$Z_N = \mathrm{Tr}[(e^{-\Delta\tau H})^N] \tag{3}$$

where $\Delta\tau = \beta/N$. By using coherent states $|c\rangle = e^{\hat{c}^\dagger c}|0\rangle$, and approximating the matrix element from time τ_j to time τ_{j+1}, where $\tau_j = j\Delta\tau$ by

$$\langle \bar{c}_{j+1}|e^{-\Delta\tau H}|c_j\rangle = e^{\alpha \bar{c}_{j+1}c_j} + O(\Delta\tau^2) \tag{4}$$

where $\alpha = (1-\Delta\tau\epsilon)$, (Fig. 1.)



(5)

show that Z_3 can be written as a "toy functional integral",

$$Z_3 = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{-(\bar{c}_3,\bar{c}_2,\bar{c}_1)\begin{bmatrix}1 & -\alpha & 0\\ 0 & 1 & -\alpha\\ \alpha & 0 & 1\end{bmatrix}\begin{pmatrix}c_3\\ c_2\\ c_1\end{pmatrix}\right\} \tag{6}$$

(b) Evaluate Z_3.

(c) Generalize the result to N time slices and obtain an expression for Z_N. What is the limiting value of your result as $N \to \infty$?