INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Questions 6. Finite Temperature. (Due Fri, Nov 19th.)

Choose two of the following questions. If you do three, the third will count for extra credit.

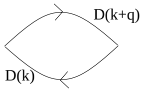
1. Use the method of complex contour integration to carry out the Matsubara sums in the following:

   (i) Derive the density of a spinless Bose Gas at finite temperature from the boson propagator $D(k) \equiv D(\mathbf{k}, \mathbf{i}\nu_\mathbf{n}) = [\mathbf{i}\nu_\mathbf{n} - \omega_\mathbf{k}]^{-1}$, where $\omega_\mathbf{k} = E_\mathbf{k} - \mu$ is the energy of a boson, measured relative to the chemical potential.

$$\rho(T) = \frac{N}{V} = V^{-1}\sum_{\mathbf{k}} \langle T b_\mathbf{k}(0^-) b^\dagger{}_\mathbf{k}(0)\rangle = -(\beta V)^{-1} \sum_{i\nu_n, \mathbf{k}} D(k) e^{i\nu_n 0^+}. \tag{1}$$

   How do you need to modify your answer to take account of Bose Einstein condensation?

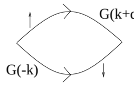
   (ii) The dynamic charge-susceptibility of a free Bose gas, i.e

$$\chi_c(q, i\nu_n) =$$


$$= T\sum_{i\nu_n}\int \frac{d^3k}{(2\pi)^3} D(q+k)D(k). \tag{2}$$

   Please analytically extend your final answer to real frequencies.

   (iii) The "pair-susceptibility" of a spin-1/2 free Fermi gas, i.e.

$$\chi_P(q, i\nu_n) =$$


$$= T\sum_{i\omega_r}\int \frac{d^3k}{(2\pi)^3} G(q+k)G(-k) \tag{3}$$

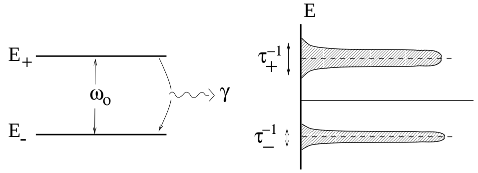
   where $G(k) \equiv G(\mathbf{k}, \mathbf{i}\omega_\mathbf{n}) = [\mathbf{i}\omega_\mathbf{n} - \epsilon_\mathbf{k}]^{-1}$. (Note the direction of the arrows: why is there no minus sign for the Fermion loop?) Show that the static pair susceptibility, $\chi_P(0)$is given by

$$\chi_P = \int \frac{d^3k}{(2\pi)^3} \frac{\tanh[\beta\epsilon_\mathbf{k}/2]}{2\epsilon_\mathbf{k}} \tag{4}$$

   Can you see that this quantity diverges at low temperatures? How does it diverge, and why ?

2. A simple model an atom with two atomic levels coupled to a radiation field is described by the Hamiltonian

$$H = H_o + H_I + H_{\text{photon}}, \tag{5}$$



   where

$$H_o = \tilde{E}_- c^\dagger{}_- c_- + \tilde{E}_+ c^\dagger{}_+ c_+ \tag{6}$$

   describes the atom, treating it as a *fermion*

$$H_I = V^{-1/2}\sum_{\vec{q}} g(\omega_{\vec{q}})\left(c^\dagger{}_+ c_- + c^\dagger{}_- c_+\right)\left[a^\dagger{}_{\vec{q}} + a_{-\vec{q}}\right] \tag{7}$$

   describes the coupling to the radiation field (V is the volume of the box enclosing the radiation) and

$$H_{photon} = \sum_{\vec{q}} \omega_{\vec{q}} a^\dagger{}_{\vec{q}} a_{\vec{q}}, \qquad (\omega_q = cq) \tag{8}$$

   is the Hamiltonian for the electromagnetic field. The "dipole" matrix element $g(\omega)$ is weak enough to be treated by second order perturbation theory and the polarization of the photon is ignored.

   (i) Calculate the self-energy $\Sigma_+(\omega)$ and $\Sigma_-(\omega)$ for an atom in the $+$ and $-$ states.

   (ii) Use the self-energy obtained above to calculate the life-times $\tau_\pm$ of the atomic states, i.e.

$$\tau_\pm^{-1} = 2\text{Im}\Sigma_\pm(\tilde{E}_\pm - i\delta). \tag{9}$$

   If the gas of atoms is non-degenerate, i.e the Fermi functions are all small compared with unity, $f(E_\pm) \sim 0$ show that

$$\begin{aligned} \tau_+^{-1} &= 2\pi|g(\omega_o)|^2 F(\omega_o)[1 + n(\omega_o)] \\ \tau_-^{-1} &= 2\pi|g(\omega_o)|^2 F(\omega_o) n(\omega_o), \end{aligned} \tag{10}$$

   where $\omega_o = \tilde{E}_+ - \tilde{E}_-$ is the separation of the atomic levels and

$$F(\omega) = \int \frac{d^3q}{(\mathbf{2}\pi)^3}\delta(\omega - \omega_q) = \frac{\omega^2}{2\pi c^3} \tag{11}$$

   is the density of state of the photons at energy ω. What do these results have to do with stimulated emission? Do your final results depend on the initial assumption that the atoms were fermions?

   (iii)Why is the decay rate of the upper state larger than the decay rate of the lower state by the factor $[1 + n(\omega_0)]/n(\omega_0)$?

3. Spectral decomposition. The dynamic spin susceptibility of a magnetic system, is defined as

$$\chi(\mathbf{q}, t_1 - t_2) = i\langle[S^-(\mathbf{q}, t_1), S^+(-\mathbf{q}, t_2)] > \theta(t_1 - t_2) \quad (12)$$

where $S^{\pm}(\mathbf{q}) = S_x(\mathbf{q}) \pm iS_y(\mathbf{q})$ are the spin raising and lowering operators at wavevector $\mathbf{q}$, i.e

$$S^{\pm}(\mathbf{q}) = \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} S^{\pm}(\mathbf{x}) \quad (13)$$

so that $S^-(\mathbf{q}) = [S^+(-\mathbf{q})]^\dagger$. The dynamic spin susceptibility determines the response of the magnetization at wavevector $\mathbf{q}$ in response to an applied magnetic field at this wavevector

$$M(\mathbf{q}, t) = \frac{\mu_B}{2} \int dt' \chi(\mathbf{q}, t - t') B(t'). \quad (14)$$

(i)Make a spectral decomposition, and show that

$$\chi(\mathbf{q}, t) = i\theta(t) \int \frac{d\omega}{\pi} \chi''(\mathbf{q}, \omega) e^{-i\omega t} \quad (15)$$

where $\chi''(\mathbf{q}, \omega)$ (often called the "power-spectrum" of spin fluctuations) is given by

$$\chi''(\mathbf{q}, \omega) = (1 - e^{-\beta\omega}) \sum_{\lambda,\zeta} e^{-\beta(E_\lambda - F)} |\langle\zeta|S^+(-\mathbf{q})|\lambda\rangle|^2 \pi\delta[\omega - (E_\zeta - E_\lambda)] \quad (16)$$

and F is the Free energy.

(ii)Fourier transform the above result to prove the integral transform

$$\chi(\mathbf{q}, \nu) = \int \frac{d\omega}{\pi} \frac{\chi''(\mathbf{q}, \omega)}{\omega - \nu - i\delta}$$

which relates $\chi(\mathbf{q}, \nu)$ and $\chi''(\mathbf{q}, \nu)$. This result is called a "Kramers Kronig" transformation and can be used to analytically extend $\chi(\vec{q}, \nu) \to \chi(\vec{q}, z)$ into the upper half complex plane. Notice how the poles of $\chi(\mathbf{q}, \nu)$ are located just below the real axis, guaranteeing causality. (Effect follows cause).

(iii) Carry out a spectral decomposition on the imaginary time response function, to show that

$$\langle TS^-(\mathbf{q}, \tau) S^+(-\mathbf{q}, 0) >= T \sum_n \chi(\mathbf{q}, i\nu_n) e^{-i\nu_n\tau}$$

where $\chi(\vec{q}, i\nu_n)$ is the analytic extension of $\chi(\vec{q}, z)$ onto the imaginary axis.

(iv)In neutron scattering experiments, the inelastic scattering cross-section is directly proportional to a spectral function called $S(\mathbf{q}, \omega)$,

$$\frac{d^2\sigma}{d\Omega d\omega} \propto S(\mathbf{q}, \omega) \quad (17)$$

where $S(\mathbf{q}, \omega)$ is the Fourier transform of a correlation function:

$$S(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle S^-(\mathbf{q}, t) S^+(-\mathbf{q}, 0)\rangle \quad (18)$$

By carrying out a spectral decomposition, show that

$$S(\mathbf{q}, \omega) = 2n(\omega)\chi''(\mathbf{q}, \omega) \quad (19)$$

This relationship, plus the one you derived in part (i) can be used to completely the dynamical spin susceptibility via inelastic neutron scattering.