**INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004**

**Solutions to Questions 6.**

1. (a) The density of bosons is given by

$$\begin{aligned}
\rho_o &= \frac{-1}{\beta V}\sum_k D(k)e^{i\nu_n 0^+} \\
&= -\frac{1}{V}\sum_{\mathbf{k}} \oint_{\text{anti-clockwise round Im axis}} \frac{dz}{2\pi i}\frac{n(z)}{(z-\omega_{\mathbf{k}})} \\
&= -\frac{1}{V}\sum_{\mathbf{k}} \oint_{\text{clockwise round } z=\omega_{\mathbf{k}}} \frac{dz}{2\pi i}\frac{n(z)}{(z-\omega_{\mathbf{k}})} \\
&= \frac{1}{V}\sum_{\mathbf{k}} n(\omega_{\mathbf{k}}) \qquad (1)
\end{aligned}$$

In the continuum limit, we may replace $V^{-1}\sum_{\mathbf{k}} \to \int \frac{d^Dk}{(2\pi)^D}$ so that

$$\rho_o = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(e^{\beta\omega_{\mathbf{k}}}-1)}$$

This expression determines the variation of the chemical potential with temperature. In $D \geq 2$, Bose-Einstein condensation may occur. This takes place when $\mu \to 0^-$, at which point, the number of Bosons in the lowest $\mathbf{k} = \mathbf{0}$ state lowest level becomes macroscopic, giving rise to a finite density of Bosons in the $\mathbf{k} = 0$ state:

$$\lim_{V\to\infty}\frac{1}{V}\left[\frac{1}{e^{-\beta\mu}-1}\right] = \rho_s(T).$$

Below the BE condensation temperature, we set $\mu = 0$, so that now

$$\rho_o = \rho_s(T) + \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(e^{\beta E_{\mathbf{k}}}-1)}$$

The first and second terms represent the density of condensed, and normal Bosons respectively. Notice that now we have $E_{\mathbf{k}}$, the actual Boson energy, in the Bose function. Notice finally that we can write the density of normal Bosons as

$$\rho_N(T) = \int dE\ N(E)\frac{1}{(e^{\beta E}-1)}$$

where $N(E)$ is the density of states. For a quadratic dispersion, $E_{\mathbf{k}} \propto k^2$, $N(E) \propto E^{(D-2)/2}$, so that in this case $\rho(T) \propto T^{D/2}$. The condensation temperature is determined by the condition that $\rho_N(T_{BE}) = \rho_o$, so that $\rho_N(T) = \rho_o(T/T_{BE})^{(D/2)}$

$$\rho_s(T) = \rho_o\left(1 - \left[\frac{T}{T_{BE}}\right]^{(D/2)}\right)$$

(b) In this case, there is no $(-1)$ for the polarization loop, because we are dealing with Bosons. The susceptibility is thus given by

$$\chi(\mathbf{q}, i\nu_n) = \int \frac{d^3k}{(2\pi)^3} \underbrace{\oint \frac{dz}{2\pi i} n(z) \frac{1}{i\nu_n + z - \omega_{\mathbf{k+q}}} \frac{1}{z - \omega_{\mathbf{k}}}}_{\text{clockwise around propagator poles}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{n(\omega_{\mathbf{q+k}}) - n(\omega_{\mathbf{k}})}{i\nu_n - \omega_{\mathbf{q+k}} + \omega_{\mathbf{k}}} \tag{2}$$

where we have substituted $n(\omega_{\mathbf{k+q}} + i\nu_n) = n(\omega_{\mathbf{k+q}})$. Analytically extending to real frequencies gives

$$\chi(\mathbf{q}, \nu) = \int \frac{d^3k}{(2\pi)^3} \frac{n(\omega_{\mathbf{q+k}}) - n(\omega_{\mathbf{k}})}{\nu - \omega_{\mathbf{q+k}} + \omega_{\mathbf{k}}} \tag{3}$$

(c) In this third example, there is no fermion loop, so there is no minus sign associated with the Matsubara sum. Converting the sum to an integral over the Fermi function, we have

$$\chi_P(\mathbf{q}, i\nu_n) = -\int \frac{d^3k}{(2\pi)^3} \underbrace{\oint \frac{dz}{2\pi i} f(z) \frac{1}{i\nu_n + z - \epsilon_{\mathbf{k+q}}} \frac{1}{-z - \epsilon_{-\mathbf{k}}}}_{\text{clockwise around propagator poles}}$$

$$= -\int \frac{d^3k}{(2\pi)^3} \left[ \frac{f(\epsilon_{\mathbf{q+k}}) - f(-\epsilon_{-\mathbf{k}})}{i\nu_n - (\epsilon_{\mathbf{q+k}} + \epsilon_{-\mathbf{k}})} \right] \tag{4}$$

Analytically extending to real frequencies then gives us

$$\chi_P(\mathbf{q}, \nu + i\delta) = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - f(\epsilon_{\mathbf{q+k}}) - f(\epsilon_{\mathbf{k}})}{\nu - (\epsilon_{\mathbf{q+k}} + \epsilon_{-\mathbf{k}}) + i\delta} \right]$$

so that the uniform pair susceptibility is

$$\chi_P = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - 2f(\epsilon_{\mathbf{k}})}{-2\epsilon_{\mathbf{k}}} \right] = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\tanh(\beta\epsilon_{\mathbf{k}}/2)}{2\epsilon_{\mathbf{k}}} \right]$$

The contribution to this integral from near the Fermi energy is given by

$$\chi_P = N(0) \int_{-\Lambda}^{\Lambda} d\epsilon \frac{\tanh(\beta\epsilon/2)}{2(\epsilon)} \sim N(0) \ln\left(\frac{\Lambda}{T}\right).$$

showing the logarithmic divergence at low temperatures.

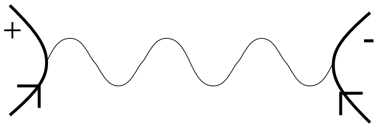
2. In the absence of the interaction, we have a free field theory. The propagators for the electrons are then

$$G_\lambda(i\omega_n) = \frac{1}{i\omega_n - \epsilon_\lambda}, \qquad (\lambda = \pm)$$

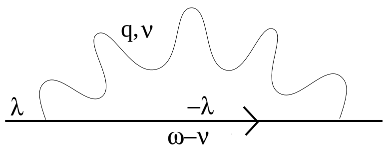
whilst the propagator for the photon is given by

$$D(\mathbf{q}, i\nu n) = \frac{2\omega_{\mathbf{q}}}{(i\nu_n)^2 - \omega_{\mathbf{q}}^2}$$

The interaction between the atom and the photon field causes transitions between the (+) and the (-) state. The electron-photon interaction is denoted by the diagram




and in leading order perturbation theory, this gives rise to the self-energy




$$= \Sigma_\lambda(\omega) = \tfrac{1}{\beta V} \textstyle\sum_{\vec{q},\nu} D(\vec{q},\nu) G_{-\lambda}(\omega - \nu)$$

Using finite temperature perturbation theory, this self-energy is given by

$$\Sigma_\pm(i\omega_n) = -T \sum_{\vec{q}\ i\nu_r} g(\omega_{\vec{q}})^2 D(\vec{q}, i\nu_r) G_\mp(i\omega_n - i\nu_r) \tag{5}$$

Carrying out the Matsubara sum as a contour integral, we find

$$\sum_{i\nu_r} \frac{1}{(i\nu_r - \omega_q)} \frac{1}{(i\omega_n - i\nu_r - E)} = \frac{1 + n(\omega_q) - f(E)}{(i\omega_n - (\omega_q + E))} \tag{6}$$

and

$$-\sum_{i\nu_r} \frac{1}{(i\nu_r + \omega_q)} \frac{1}{(i\omega_n - i\nu_r - E)} = \frac{n(\omega_q) + f(E)}{(i\omega_n - (E - \omega_q))} \tag{7}$$

so that the full self-energy is

$$\Sigma_\pm(i\omega_n) = \int d\nu g^2(\nu) F(\nu) \left[ \frac{1 + n(\nu) - f(\epsilon_\mp)}{(i\omega_n - (\nu + \epsilon_\mp))} + \frac{n(\nu) + f(\epsilon_\mp)}{(i\omega_n - (\epsilon_\mp - \nu))} \right] \tag{8}$$

(b.) Analytically extending the answer from (a) into the complex plane, we obtain

$$\Sigma_\pm(z) = \int d\nu g^2(\nu) F(\nu) \left[ \frac{1 + n(\nu) - f(\epsilon_\mp)}{(z - (\nu + \epsilon_\mp))} + \frac{n(\nu) + f(\epsilon_\mp)}{(z - (\epsilon_\mp - \nu))} \right] \tag{9}$$

Taking the imaginary part gives

$$\mathrm{Im}\Sigma_\pm(\omega - i\delta) = \pi \left\{ \left( g^2(\nu) F(\nu) [1 + n(\nu) - f(\epsilon_\mp)] \right)_{\nu = \omega - \epsilon_\mp} + \left( g^2(\nu) F(\nu) [n(\nu) + f(\epsilon_\mp)] \right)_{\nu = \epsilon_\mp - \omega} \right\} \tag{10}$$

The life-time of the (+) and (-) states are determined by the imaginary part of the self-energy, evaluated at the position of the quasiparticle pole, which gives

$$\Gamma_+ \;=\; 2\pi g^2(\omega_o) F(\omega_o) [1 + n(\omega_o) - f(\epsilon_-)]$$

$$\Gamma_- = 2\pi g^2(\omega_o)F(\omega_o)[n(\omega_o) + f(\epsilon_+)] \tag{11}$$

(c) In the limit that $f(\epsilon_-)$ and $f(\epsilon_+) \to 0$, the decay rates become

$$\begin{aligned} \Gamma_+ &= 2\pi g^2(\omega_o)F(\omega_o)[1 + n(\omega_o)] \\ \Gamma_- &= 2\pi g^2(\omega_o)F(\omega_o)[n(\omega_o)] \end{aligned} \tag{12}$$

Even in the $T \to 0$ limit, the upper state has a lifetime, but at finite temperatures the decay rate is enhanced by stimulated emission. The second result is recognized as the decay rate due to photon absorption, and this goes to zero in the $T \to 0$ limit.

(d) One can understand the ratio of decay rates in two ways. First, note that the two decay rates satisfy "detailed balance". In thermal equilibrium, the population of atoms in the ground-state and in the excited state must be stationary. In the limit where $f_+$ and $f_- \to 0$, the ratio between the number of atoms in the excited state and ground-state is $N_+/N_- = (f_+/f_-) \sim e^{-\beta\omega_o}$. Thus the ratio of transition rates is

$$\frac{N_+\Gamma_+}{N_-\Gamma_-} = e^{-\beta\omega_o}\frac{n(\omega_o)+1}{n(\omega_o)} = 1 \tag{13}$$

showing that the decay rates satisfy detailed balance in this limit.

Finally, note that $\Gamma_+$ involves the emission of photons whereas $\Gamma_-$ involves the absorption of photons. The matrix element for the emission of a photon into a n-photon state is proportional to

$$\langle n-1|a|n\rangle = \sqrt{1+n}, \tag{14}$$

whereas the matrix element for the absorption of a photon is proportional to

$$\langle n+1|a^\dagger|n\rangle = \sqrt{n}, \tag{15}$$

Since the decay rates are proportional to the square of the matrix elements, the ratio of $\Gamma_+/\Gamma_-$ is proportional to the $(1+n)/n$ where $n = n(\omega_0)$ is the average number of photons at energy $\omega_o$ in thermal equilibrium.

3. (i) We make the spectral decomposition as follows,

$$\begin{aligned} \chi(\mathbf{q}, t_1 - t_2) &= i\theta(t_1 - t_2)\sum_\lambda e^{-\beta(E_\lambda - F)}\langle\lambda|[S^-(\mathbf{q}, t_1),\ S^+(-\mathbf{q}, t_2)]|\lambda\rangle \\ &= i\theta(t_1 - t_2)\sum_\lambda e^{-\beta(E_\lambda - F)}\underbrace{\langle\lambda|S^-(\mathbf{q}, t_1)|\zeta\rangle\langle\zeta|\ S^+(-\mathbf{q}, t_2)|\lambda\rangle}_{\langle\zeta|S^+(\mathbf{q})|\lambda\rangle|^2 e^{-i(E_\zeta - E_\lambda)(t_1 - t_2)}}(1 - e^{-\beta(E_\zeta - E_\lambda)}) \end{aligned} \tag{16}$$

where in the bracketed term, we have identified

$$\langle\lambda|S^-(\mathbf{q})|\zeta\rangle = \langle\zeta|S^+(-\mathbf{q})|\lambda\rangle^* \tag{17}$$

Introducing the spectral function

$$\chi''(\mathbf{q}, \omega) = (1 - e^{-\beta\omega})\sum_{\lambda,\zeta} e^{-\beta(E_\lambda - F)}|\langle\zeta|S^+(-\mathbf{q})|\lambda\rangle|^2\pi\delta[\omega - (E_\zeta - E_\lambda)] \tag{18}$$

we may rewrite this in the form

$$\chi(\mathbf{q}, t_1 - t_2) = i\theta(t_1 - t_2) \int \frac{d\omega'}{\pi} \chi''(\mathbf{q}, \omega') e^{-i\omega'(t_1 - t_2)} \tag{19}$$

(ii) We now Fourier transform the dynamic susceptibility with respect to time, carrying out

$$\chi(\mathbf{q}, \omega) = \int dt \chi(\mathbf{q}, t) e^{i\omega t} \tag{20}$$

Inserting the the result

$$i \int_0^\infty dt e^{[i(\omega - \omega') - \delta]t} = \frac{1}{\omega' - (\omega + i\delta)} \tag{21}$$

into (18), we obtain

$$\chi(\mathbf{q}, \omega) = \int \frac{d\omega'}{\pi} \frac{1}{\omega' - (\omega + i\delta)} \chi''(\mathbf{q}, \omega') \tag{22}$$

(iii) First, note that the correct Green function is the time ordered Green function $\mathcal{G}(\mathbf{q}, \tau) = \langle T S^-(\mathbf{q}, \tau) S^+(-\mathbf{q}, 0) >$. Carrying out the spectral decomposition of this correlation function for the case $\tau > 0$, we obtain

$$\langle S^-(\mathbf{q}, \tau) S^+(\mathbf{q}, 0) \rangle = \sum_{\lambda,\zeta} e^{-\beta(E_\zeta - F)} \langle \zeta | S^-(\mathbf{q}) | \lambda \rangle \langle \lambda | S^+(\mathbf{q}) | \zeta \rangle e^{-(E_\zeta - E_\lambda)\tau} \tag{23}$$

Repeating the same derivation for $\mathcal{G}(\mathbf{q}, \tau - \beta) = \langle S^+(\mathbf{q}, 0) S^-(\mathbf{q}, \tau - \beta) \rangle$, we find that $\mathcal{G}(\mathbf{q}, \tau - \beta) = \mathcal{G}(\mathbf{q}, \tau)$, so this function is periodic in time, and must have the form

$$\mathcal{G}(\mathbf{q}\tau) = T \sum_n \mathcal{G}(\mathbf{q}, i\nu_n) e^{-i\nu_n \tau} \tag{24}$$

where

$$\begin{aligned}
\mathcal{G}(\mathbf{q}, i\nu_n) &= \int_0^\beta \mathcal{G}(\mathbf{q}, \tau) e^{i\nu_n \tau} \\
&= \sum_{\lambda,\zeta} e^{-\beta(E_\zeta - F)} \langle \zeta | S^-(\mathbf{q}) | \lambda \rangle \langle \lambda | S^+(\mathbf{q}) | \zeta \rangle \frac{e^{-(E_\zeta - E_\lambda)\beta} - 1}{i\nu_n - (E_\zeta - E_\lambda)} \\
&= \int \frac{d\omega}{\pi} \frac{1}{\omega - i\nu_n} \chi''(\mathbf{q}, \omega) \\
&= \chi(\mathbf{q}, i\nu_n).
\end{aligned} \tag{25}$$

(iv) Carrying out the spectral decomposition of the correlation function, we obtain

$$\begin{aligned}
\langle S^-(\mathbf{q}, t) S^+(-\mathbf{q}, 0) \rangle &= \sum_{\lambda,\zeta} e^{-\beta(E_\lambda - F)} \langle \lambda | S^-(\mathbf{q}) | \zeta \rangle \langle \zeta | S^+(-\mathbf{q}) | \lambda \rangle e^{-i(E_\zeta - E_\lambda)t} \\
&= \int \frac{d\omega}{\pi} e^{-i\omega t} \underbrace{\sum_{\lambda,\zeta} e^{-\beta(E_\lambda - F)} \pi \delta(\omega - (E_\zeta - E_\lambda)) |\langle \zeta | S^+(-\mathbf{q}) | \lambda \rangle|^2}_{(1 - e^{-\beta\omega})^{-1} \chi''(\mathbf{q}, \omega)}
\end{aligned}$$

$$= \int \frac{d\omega}{2\pi} \frac{2}{(1 - e^{-\beta\omega})} \chi''(\mathbf{q}, \omega) e^{-i\omega t} \qquad (26)$$

Carrying out the Fourier transform of this result, then yields

$$S(\mathbf{q}, \omega) = 2[1 + n(\omega)]\chi''(\mathbf{q}, \omega). \qquad (27)$$