INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2002

Midterm Questions 5. (Due Weds, Nov 12)

Do at least two of the following. Anything beyond two answers will be treated as extra credit.

1. A particle with $S = 1/2$ is placed in a large magnetic field $\vec{B} = (B_1 \cos(\omega t), B_1 \sin(\omega t), B_o)$, where $B_o >> B_1$.

   (a) Treating the oscillating part of the Hamiltonian as the interaction, write down the Schrödinger equation in the interaction representation. (i.e write out explicitly, the differential equation $H_I(t)\psi_I(t) = i\hbar\partial\psi_I(t)/\partial t$)

   (b) By explicitly solving the Schrödinger equation obtained in (b), evaluate the time evolution operator in the interaction representation (i.e Find $U(t) = \mathrm{T}\exp\left[-iH_{int}(t')dt'\right]$.)

   (c) If the particle starts out at time $t = 0$ in the state $S_z = -\frac{1}{2}$, what is the probability it is in this state at time t ?

2. Consider a gas of particles with interaction

$$\hat{V} = 1/2 \sum_{\vec{k}\vec{k}'\vec{q}\sigma\sigma'} V_q c^{\dagger}{}_{\vec{k}-\vec{q}\sigma} c^{\dagger}{}_{\vec{k}'+\vec{q}\sigma'} c_{\vec{k}'\sigma'} c_{\vec{k}\sigma}$$

   (a) Let $|\phi_0\rangle = \prod_{|k|<k_F,\sigma} c^{\dagger}{}_{\vec{k}\sigma}|0\rangle$ represent the filled Fermi sea, i.e. the ground state of the non interacting problem. Use Wick's theorem to evaluate an expression for the expectation value of the interaction energy $\langle\phi_0|\hat{V}|\phi_0\rangle$ in the non-interacting ground state. Give a physical interpretation of the two terms that arise and draw the corresponding Feynman diagrams. You may use the result

$$\langle\phi|c^{\dagger}{}_{\vec{k}\sigma} c_{\vec{k}'\sigma'}|\phi\rangle = \delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}\theta(k_F - k) \tag{1}$$

   (b) Draw the all the Feynman diagrams corresponding to the second order corrections to the ground-state energy.

   (c) Discuss how to convert one or more of the above Feynman diagrams into a mathematical expression, using the Feynman rules.

3. Suppose we turn on the interaction adiabatically, writing $V_{int}(\lambda) = \lambda\hat{V}$. It is possible to relate the change in the Free energy ΔF to the interaction energy averaged over interaction strength with weight $1/\lambda$. Prove this using the following outline.

   (a) Consider the Free energy of the system, which is given by $F(\lambda) = -T \ln \mathrm{Tr}[e^{-\beta[H_o+\lambda V]}]$. First show by differentiating this result that

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{\lambda}\langle V_{int}(\lambda)\rangle \tag{2}$$

   (b) Use this result to reason that the change in the free energy as a result of turning on the interaction is given by

$$\Delta F = \int_0^1 \frac{d\lambda}{\lambda}\langle V_{int}(\lambda)\rangle \tag{3}$$

By taking the zero temperature limit of this expression, we can reason that

$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \phi(\lambda) | V_{int}(\lambda) | \phi(\lambda) \rangle \tag{4}$$

where $|\phi(\lambda)\rangle$ is the ground-state for $H(\lambda) = H_0 + \lambda \hat{V}$. In otherwords, the change in energy is equal to the interaction energy, averaged over the interaction strength, with weight $1/\lambda$.

(c) If the interaction energy has an expansion $\langle V_{int}(\lambda) \rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, what is the corresponding expression for the change in the ground-state energy in increasing λ to full strength $\lambda = 1$?

(d) Why is the change in the ground-state energy <u>not</u> equal to the change in the interaction energy? (Hint- as the interaction increases the Hamiltonian is changing. What else is changing?)

4. (Alternative derivation of bosonic generating functional.) (i) Consider the forced Harmonic oscillator

$$H(t) = \omega b^\dagger b + \bar{z}(t) b + b^\dagger z(t) \tag{5}$$

where $z(t)$ and $\bar{z}(t)$ are arbitrary, independent functions of time. Consider the S-matrix

$$S[z, \bar{z}] = \langle 0 | \mathrm{T} \hat{S}(\infty, -\infty) | 0 \rangle = \langle 0 | \mathrm{T} \exp\left(-i \int_{-\infty}^{\infty} dt [\bar{z}(t) b(t) + b^\dagger(t) z(t)] \right) |0\rangle, \tag{6}$$

where $\hat{b}(t)$ denotes $\hat{b}$ in the interaction representation. Consider changing the function $\bar{z}(t)$ by an infinitesimal amount

$$\bar{z}(t) \rightarrow \bar{z}(t) + \Delta\bar{z}(t_o)\delta(t - t_o), \tag{7}$$

The quantity

$$\lim_{\Delta\bar{z}(t_o)\to 0} \frac{\Delta S[z,\bar{z}]}{\Delta\bar{z}(t_o)} = \frac{\delta S[z,\bar{z}]}{\delta\bar{z}(t_o)}$$

is called the "functional derivative" of S with respect to $\bar{z}$. Using the Gell-Man Lowe formula $\langle \psi(t) | b | \psi(t) \rangle = \frac{\langle 0 | T\hat{S}(\infty,-\infty) b(t) | 0 \rangle}{\langle 0 | T\hat{S}(\infty,-\infty) | 0 \rangle}$ prove the following identity

$$i\delta \mathrm{ln} S[z,\bar{z}]/\delta\bar{z}(t) \equiv \tilde{b}(t) = \langle \hat{b}(t) \rangle = \langle \psi(t) | \hat{b} | \psi(t) \rangle. \tag{8}$$

(ii) Use the equation of motion to show that

$$\frac{\partial}{\partial t}\tilde{b}(t) = i\langle [H(t), \hat{b}(t)] \rangle = -i[\epsilon \tilde{b}(t) + z(t)].$$

(iii) Solve the above differential equation to show that

$$\tilde{b}(t) = \int_{-\infty}^{\infty} G(t - t') z(t') \tag{9}$$

where $G(t - t') = -i\langle 0 | T[b(t) b^\dagger(t')] | 0 \rangle$ is the free Green's function for the harmonic oscillator.

(iv) Use (iii) and (i) together to obtain the fundamental result

$$S[z,\bar{z}] = \exp\left[-i \int_{-\infty}^{\infty} dt dt' \bar{z}(t) G(t - t') z(t') \right] \tag{10}$$

5. (A good answer on this will be given generous extra credit.) Look up the original paper by Landau on the Fermi liquid and the experimental paper by Fairbanks et al. on the normal state of He_3 that he references. Why is $He - 3$ the ideal Fermi liquid? Discuss the two papers, identifying the key experimental observations which could not be understood, and the way in which Landau accounted for the discrepancies.