INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Questions 4. (Due Weds, Oct 27th)

Choose three out of the following four questions. If you have time, do the fourth for extra credit.

1. (a) Consider the non-interacting Hubbard model for next nearest neighbor hopping on a two dimensional lattice

$$H - \mu N = -t \sum_{j,\hat{a}=\hat{x},\hat{y},\sigma} [c^\dagger{}_{j+\hat{a}\sigma} c_{j\sigma} + \mathrm{H.c}] - \mu \sum_{j\sigma} c^\dagger{}_{j\sigma} c_{j\sigma}$$

where $n_{j\sigma} = c^\dagger{}_{j\sigma} c_{j\sigma}$ represents the number of electrons of spin component $\sigma = \pm 1/2$ at site j. Show that the dispersion of the electrons in the absence of interactions is given by

$$\epsilon(\vec{k}) = -2t(\cos k_x a + \cos k_y a) - \mu$$

where a is the distance between sites, and $\vec{k} = (k_x, k_y)$ is the wavevector.

(b) Derive the relation between the number of electrons per site n_e and the area of the Fermi surface.

(c) Sketch the Fermi surface when

i. $n_e < 1$.

ii. "half filling" where $n_e = 1$

(d) The corresponding interacting Hubbard model, with an interaction term $U n_\uparrow n_\downarrow$ at each site describes a class of material called "Mott insulators", which includes the mother compounds for high temperature superconductors. What special feature of the Fermi surface at half-filling makes the non-interacting ground-state unstable to spin density wave formation and the development of a gap around the Fermi surface ?

(e) Derive the dispersion for the case when, in the one-particle Hamiltonian there is an additional next-nearest neighbor hopping matrix element of strength across the diagonal, $-t'$. (Hint: use the Fourier transform of $t(R)$, given by $t(\vec{k}) = \sum_{\vec{R}} t(\vec{R}) e^{-i\vec{k}\cdot\vec{R}}$). How does this affect the dispersion at half filling?

2. Consider an atom trap where the confining potential is given by a harmonic potential

$$V(x, y, z) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$$

so that the quantized kinetic energy of an atom in the trap is

$$\begin{aligned} E_{lmn} &= \hbar\omega(l + \frac{3}{2}) \\ l &= n_1 + n_2 + n_3 \end{aligned} \qquad (1)$$

where $n_{1,2,3} \geq 0$ are non-negative integers.

(a) Show that in thermal equilibtrium, the total number of particles in the trap is given by

$$N = \frac{z}{1-z} + \sum_{l \neq 0} \frac{1}{z^{-1} e^{\beta\hbar\omega l} - 1}$$

where $z = e^{\beta\mu^*}$, $\mu^* = \mu - \frac{3}{2}\hbar\omega$ and the sum over l, denotes a sum over all values of $n_{1,2,3}$. What is the interpretation of the first term? Why must μ^* be negative? What happens to z when condensation develops?

(b) Show that below the Bose Einstein condensation temperature,

$$N = N_0(T) + \sum_{k=1}^{\infty} \left[\left(\frac{1}{1 - e^{-xk}} \right)^3 - 1 \right], \qquad (x = \beta\hbar\omega)$$

where $N_0(T)$ is the number of atoms condensed in the lowest energy state. (Hint, Taylor expand the second term in (a) and then invert the order of summation.)

(c) Show that when x is small,

$$N = N_o(T) + \frac{T^3}{(\hbar\omega)^3}\zeta(3) + \frac{T^2}{(\hbar\omega)^2}\zeta(2) + O(T)$$

where $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$. Notice that if the first term is of order N, the second term is of order $N^{2/3} << N$, if N is large.

(d) Calculate an expression for the Bose Einstein condensation temperature T_{BE} of the trap, and show that to a good approximation,

$$\frac{N_o(T)}{N} = 1 - \left(\frac{T}{T_{BE}} \right)^3.$$

(e) Qualitatively: what do you expect to happen to the density profile across the trap, once the atoms start to condense?

3. Suppose we turn on the interaction adiabatically, writing $V_{int}(\lambda) = \lambda\hat{V}$. It is possible to relate the change in the Free energy ΔF to the interaction energy averaged over interaction strength with weight $1/\lambda$. Prove this using the following outline.

(a) Consider the Free energy of the system, which is given by $F(\lambda) = -T \ln \mathrm{Tr}[e^{-\beta[H_o + \lambda V]}]$. First show by differentiating this result that

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{\lambda}\langle V_{int}(\lambda)\rangle \qquad (2)$$

(b) Use this result to reason that the change in the free energy as a result of turning on the interaction is given by

$$\Delta F = \int_0^1 \frac{d\lambda}{\lambda} \langle V_{int}(\lambda)\rangle \qquad (3)$$

By taking the zero temperature limit of this expression, we can reason that

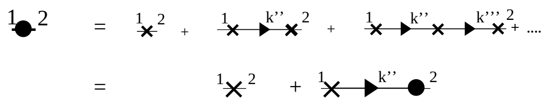
$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \phi(\lambda) | V_{int}(\lambda) | \phi(\lambda) \rangle \tag{4}$$

where $|\phi(\lambda)\rangle$ is the ground-state for $H(\lambda) = H_0 + \lambda \hat{V}$. In otherwords, the change in energy is equal to the interaction energy, averaged over the interaction strength, with weight $1/\lambda$.

(c) If the interaction energy has an expansion $\langle V_{int}(\lambda) \rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \ldots$, what is the corresponding expression for the change in the ground-state energy in increasing λ to full strength $\lambda = 1$?

(d) Why is the change in the ground-state energy not equal to the change in the interaction energy? (Hint- as the interaction increases- what is changing, other than the Hamiltonian?)

4. Construct the diagram technique for the one-particle scattering amplitude on the potential $U(x)$.



(a) What is denoted by the crosses and the solid line?

(b) Show that the electron Green's function has the form

$$G_{\vec{k},\vec{k}'}(E) = G^{(0)}_{\vec{k}}(E)\delta_{\vec{k},\vec{k}'} + G^{(0)}_{\vec{k}}(E) t_{\vec{k},\ \vec{k}'}(E) G^{(0)}_{\vec{k}}(E) \tag{5}$$

where the scattering amplitude satisfies the integral equation

$$t_{\vec{k},\vec{k}'}(E) = U(\vec{k} - \vec{k}') + \int \frac{d^d q}{(2\pi)^d} \frac{U(\vec{k} - \vec{q})}{E - E(q) + i\delta} t_{\vec{q},\vec{k}'}(E) \tag{6}$$

(c) Evaluate the t-matrix for a delta-function potential $U(x) = U\delta^{(d)}(x)$. (Hint: the t-matrix is momentum independent).

(d) Show that in dimensions $d \leq 2$, a bound-state forms for arbitrarilly weak scattering potential. (Hint: this means checking to see if there are negative energy poles in the t-matrix)