INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2003

Answers to Questions 4. (Mon, Nov 2)

1. (a) If we insert the Fourier transformed operator

$$c_{j\sigma} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} c_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{R}_j} \tag{1}$$

into the kinetic part of the Hamiltonian

$$H = -t \sum_{\hat{a}=\hat{x},\hat{y},j\sigma} [c^\dagger{}_{j+\hat{a}\sigma} c_{j\sigma} + \text{H.c}] \tag{2}$$

we obtain

$$\begin{aligned} H &= -t \sum_{\vec{k},\vec{k}'\sigma} [c^\dagger{}_{\vec{k}\sigma} c_{\vec{k}'\sigma} e^{i(k_x+k_y)} \overbrace{\frac{1}{N}\sum_j e^{iR_j\cdot(\vec{k}-\vec{k}')}}^{\delta_{\vec{k},\vec{k}'}} + \text{H.c}] \\ &= \sum_{\vec{k}\sigma} [-2t(\cos k_x + \cos k_y)]\, c^\dagger{}_{\vec{k}\sigma} c_{\vec{k}\sigma}. \end{aligned}$$

Combining this with the chemical potential term $-\mu N = -\mu \sum_{\vec{k}\sigma} c^\dagger{}_{\vec{k}\sigma} c_{\vec{k}\sigma}$, we obtain

$$H = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}} c^\dagger{}_{\vec{k}\sigma} c_{\vec{k}\sigma} \tag{3}$$

where $\epsilon(\vec{k}) = -2t(\cos k_x a + \cos k_y a) - \mu$.

(b) The total number of particles is given by

$$N_e = \sum_{\vec{k}\sigma} \langle c^\dagger{}_{\vec{k}\sigma} c_{\vec{k}\sigma} \rangle \tag{4}$$

In the thermodynamic limit, we can replace $\sum_{\vec{k}} \rightarrow V \int_{\vec{k}}$, where $A = Na^2$ is the area of the system. Thus

$$N_e = Na^2 \sum_\sigma \int \frac{d^2k}{(2\pi)^2} n_{\vec{k}\sigma} \tag{5}$$

Now in the ground-state, $n_{\vec{k}\sigma} = \theta(-\epsilon_{\vec{k}})$ so that

$$N_e = 2Na^2 \int \frac{d^2k}{(2\pi)^2} \theta(-\epsilon) \tag{6}$$

We can identify the integral as $A_{FS}/(2\pi)^2$, where A_{FS} is the area of the Fermi surface. Thus the number of particles per unit area is

$$n_e = \frac{N_e}{Na^2} N_e = 2\frac{A_{FS}}{(2\pi)^2}. \tag{7}$$

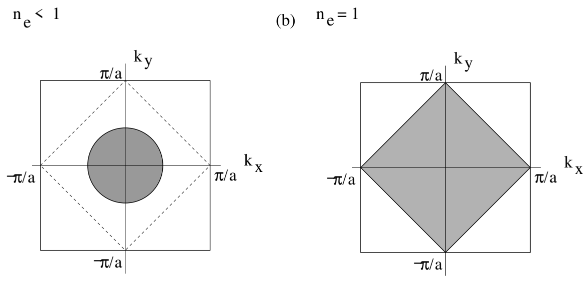

Figure 1:

(c) (a) When $\mu < 0$, the band is less than half filled, with $n_e < 1$. In this case, the Fermi surface is centered around $k = 0$ and for small filling, approximates a circle. (See below) (b) When $\mu = 0$, the band is "half filled" and $n_e = 1$. The Fermi surface is now a square, rotated through 45^0, occupying one half of the Brillouin zone, as shown in Fig. 1.

(d) When the band is half filled, the edges of the Fermi surface are parallel, and are said to be "nested". In this situation, the susceptibility of the metal towards the formation of a spin-density wave is logarithmically enhanced, which leads to a spin-density wave instability the moment a finite Hubbard U interaction is introduced into the system.

(e) If we introduce a second neighbor coupling t' across the diagonal, then the Fourier transform of the bond-hopping $t(\vec{R})$ becomes

$$t_{\vec{k}} = -2t(c_x + c_y) + 4t' c_x c_y$$

where $c_{x,y} \equiv \cos k_{x,y} a$. The dispersion is now

$$\epsilon_{\vec{k}} = t_{\vec{k}} = -2t(c_x + c_y) + 4t' c_x c_y - \mu \tag{8}$$

The next-neighbor coupling suppresses hopping along the diagonals, causing the Fermi surface to bulge along the diagonals at half-filling.

2. (a) The number of particles in the "$l-th$" level is given by the Einstein formula

$$n_l = \frac{1}{e^{\beta E_l} - 1} = \frac{1}{e^{\beta\hbar\omega_l} z^{-1} - 1} \tag{9}$$

where $z = e^{\beta(\mu - \hbar\omega 3/2}$. The total number of particles is then

$$N_e = \sum_l n_l = n_{l=0} + \sum_{l\neq 0} \frac{1}{e^{\beta\hbar l\omega} z^{-1} - 1} \quad = \quad N_0(T) + \sum_{l\neq 0} n_l$$

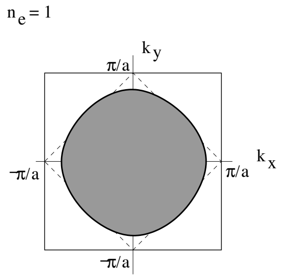

Figure 2:

$$ \tag{11} $$

where $N_0 \equiv n_0 = \frac{z}{(1-z)}$. Beneath the Bose-Einstein condensation temperature $z \to 1$ and the number of bosons in the $l = 0$ state becomes quasi-macroscopic.