INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Answers to Questions III. Oct. 17th

Here is an outline of the solutions.

1. (a) We begin by writing down the partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu N)}\right] \tag{1}$$

Differentiating w.r.t. μ, we have

$$\frac{\partial Z}{\partial \mu} = \beta \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu N)}\hat{N}\right] \tag{2}$$

So putting $F = (-1/\beta)\ln Z$, we have

$$-\frac{\partial F}{\partial \mu} = \frac{1}{\beta Z}\frac{\partial Z}{\partial \mu} = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu N)}\hat{N}\right] = \langle \hat{N} \rangle \tag{3}$$

(b) If we take

$$H - \mu N = (\epsilon - \mu)a^\dagger a \tag{4}$$

then the partition function is

$$Z = \begin{cases} \sum_{n=0}^{\infty} e^{-\beta n(\epsilon-\mu)} = (1 - e^{-\beta(\epsilon-\mu)})^{-1} & \text{(Boson)} \\ 1 + e^{-\beta(\epsilon-\mu)} & \text{(Fermion)} \end{cases} \tag{5}$$

Taking the logarithm of these expressions, we obtain

$$F = -k_B T \ln Z = \pm k_B T \ln[1 \mp e^{-\beta(\epsilon-\mu)}] \tag{6}$$

where the upper sign refers to bosons, the lower, to fermions. Finally, taking the derivative of this expression w.r.t. μ yields

$$-\frac{\partial F}{\partial \mu} = \langle \hat{n} \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} \mp 1} \tag{7}$$

2. (a) We may estimate the Bose Einstein transition temperature from

$$T_{BE} = \frac{3.31}{k_B}\left(\frac{\hbar^2 n^{2/3}}{m}\right) = \frac{3.31}{1.38 \times 10^{-23}}\left(\frac{\hbar^2 (10^{21} m^{-3})^{2/3}}{23 m_p}\right) \approx 6.9\mu K.$$

These tiny temperatures are attained by "evaporative cooling" . Sodium atoms are held in a "magneto-optic" trap. Radio waves are used to "evaporate" the most energetic atoms in the trap, leaving behind the cold ones.

(b) In Helium-4, we may estimate the Bose Einstein transition temperature as

$$T_{BE} = \frac{3.31}{k_B}\left(\frac{\hbar^2 n^{2/3}}{m_{He}}\right) = \frac{3.31}{1.38\times 10^{-23}}\left(\frac{\hbar^2((122/(4m_p)))^{2/3}}{4m_p}\right) \approx 2.76K.$$

The actual condensation temperature is 2.21K. The difference in condensation temperatures is due to the repulsive interaction between atoms.

(a) If the interaction has the form

$$V(r) = \begin{cases} U, & (r < R), \\ 0, & (r > R), \end{cases} \tag{8}$$

then in second-quantized form, the interaction Hamiltonian is

$$V = \frac{U}{2}\sum_{\sigma,\sigma'}\int d^3x \int_{|\vec{x}'-\vec{x}|<R} d^3x' [\psi^\dagger{}_\sigma(x)\psi^\dagger{}_{\sigma'}(x')\psi_{\sigma'}(x')\psi_\sigma(x)]. \tag{9}$$

(ii) Inverting the Fourier transform, we have $c_{\vec{k}\sigma} = \int d^3x\psi_\sigma(\vec{x})e^{-i\vec{k}\cdot\vec{x}}$, so that

$$\begin{aligned} [c_{\vec{k}\sigma},\ c^\dagger{}_{\vec{k}'\sigma'}]_\pm &= \int d^3x d^3x' [\psi_\sigma(x),\ \psi^\dagger{}_{\sigma'}(x')]_\pm e^{-i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')} \\ &= \delta_{\sigma\ \sigma'}\int d^3x e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} \\ &= \delta_{\sigma\ \sigma'}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'). \end{aligned} \tag{10}$$

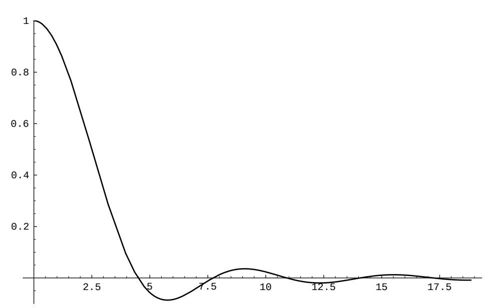

Figure 1:

(iii) In momentum space, we may write

$$V = \frac{1}{2}\int \frac{d^3k d^3k' d^3q}{(2\pi)^9} V(q)\left[c^\dagger{}_{\vec{k}+\vec{q}\sigma} c^\dagger{}_{\vec{k}'-\vec{q}\sigma'} c_{\vec{k}'\sigma'} c^\dagger{}_{\vec{k}\sigma}\right], \tag{11}$$

where

$$V(\vec{q}) = \int d^3x V(\vec{x}) e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi U}{q} \int_0^R dr r \sin(qr) = \left(\frac{4\pi R^3 U}{3}\right) F(qR) \tag{12}$$

and

$$F(x) = \frac{3}{x^2} \left[\frac{\sin x}{x} - \cos x\right]. \tag{13}$$

The form of the interaction in momentum space is sketched above. The hard core in real space is manifested as a long-range oscillatory component in momentum space.