INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2004

Answers to Questions II. Oct. 4th

Here is an outline of the solutions.

1. (i) Expanding $|\psi\rangle = |1111100\ldots\rangle = c^\dagger{}_5 c^\dagger{}_4 c^\dagger{}_3 c^\dagger{}_2 c^\dagger{}_1 |0\rangle$ we obtain

$$\begin{aligned}
c^\dagger{}_3 c_6 c_4 c_6{}^\dagger c_3 |\psi\rangle &= c^\dagger{}_3 c_6 c_4 c_6{}^\dagger c_3 c^\dagger{}_5 c^\dagger{}_4 c^\dagger{}_3 c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= (-1) c^\dagger{}_5 c^\dagger{}_3 c^\dagger{}_2 c^\dagger{}_1 |0\rangle \\
&= -|1110100\ldots\rangle.
\end{aligned}$$

(ii) We may write

$$\begin{aligned}
|110110010\ldots &= c^\dagger{}_8 |11011000\ldots\rangle \\
&= c^\dagger{}_8 c_3 |11111000\ldots\rangle.
\end{aligned} \tag{1}$$

This state can be interpreted as the creation of an electron and "hole" in states 8 and 3 respectively.

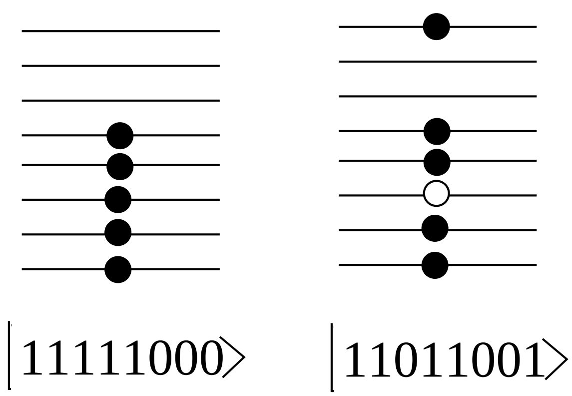

Figure 1:

(iii) To calculate $\langle\psi|\hat{N}|\psi\rangle$, where $\psi = A|100\ldots\rangle + B|111000\ldots\rangle$, note that

$$\hat{N}|\psi\rangle = A|100\ldots\rangle + 3B|111000\ldots\rangle, \tag{2}$$

so that $\langle\psi|\hat{N}|\psi\rangle[|A|^2 + 3|B|^2]$.

2. (i) We need to confirm that $\{c_1,\ c^\dagger{}_1\} = \{c_1,\ c^\dagger{}_1\} = 1$ and also $\{c_1,\ c_2\} = \{c_2,\ c_1\} = 0$. Substituting for c_1 and c_2, we obtain

$$\{c_1,\ c_2\} = \{ua_1 + va^\dagger{}_2, -va^\dagger{}_1 + ua_2\} = -uv\{a_1, a_1{}^\dagger\} + vu\{a^\dagger{}_2, a_2\} = 0, \tag{3}$$

and

$$\{c_1,\ c^\dagger{}_2\} = \{ua_1 + va^\dagger{}_2, u^* a^\dagger{}_1 + v^* a_2\} = |u|^2\{a_1, a_1{}^\dagger\} + |v|^2\{a^\dagger{}_2, a_2\} = 1, \tag{4}$$

provided $|u|^2 + |v|^2 = 1$.

(ii) Consider $H = \omega[c^\dagger{}_1 c_1 - c_2 c^\dagger{}_2]$, then if

$$\begin{pmatrix} c_1 \\ c^\dagger{}_2 \end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \begin{pmatrix} a_1 \\ a^\dagger{}_2 \end{pmatrix} = U \begin{pmatrix} a_1 \\ a^\dagger{}_2 \end{pmatrix}, \tag{5}$$

where we note U is a unitary transformation, we may re-write H as

$$H = (c_1{}^\dagger, c_2) \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2{}^\dagger \end{pmatrix}$$

so that using (5)

$$\begin{aligned} H &= (a_1{}^\dagger, a_2) U^\dagger \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} U \begin{pmatrix} a_1 \\ a_2{}^\dagger \end{pmatrix} \\ &= (a_1{}^\dagger, a_2) \begin{pmatrix} \epsilon & \Delta \\ \Delta^* & -\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2{}^\dagger \end{pmatrix} \\ &= \epsilon[a^\dagger{}_1 a_1 - a_2 a^\dagger{}_2] + [\Delta a^\dagger{}_1 a^\dagger{}_2 + \text{H.c}] \end{aligned} \tag{6}$$

where

$$\begin{aligned} \epsilon &= \omega(|u|^2 - |v|^2), \\ \Delta &= \omega 2u^* v. \end{aligned} \tag{7}$$

Squaring both expressions and adding the results, we obtain $\omega =(\epsilon^2 + \Delta^2)^{\frac{1}{2}}$ and

$$|u|^2 = \frac{1}{2}\left(1 + \frac{\epsilon}{\omega}\right), \qquad |v|^2 = \frac{1}{2}\left(1 - \frac{\epsilon}{\omega}\right), \tag{8}$$

(iii) The ground-state is annihilated by both c_1 and c_2, so that if $H = \omega[c^\dagger{}_1 c_1 + c^\dagger{}_2 c_2 - 1]$, the ground-state energy is $E_o = -\omega = -(\epsilon^2 + \Delta^2)^{\frac{1}{2}}$.

3. Let us write our starting Hamiltonian in the form

$$H = -\sum_j \left\{ \frac{J_x + J_y}{4} \left(S^+_{j+1} S^-_j + \text{H.c.} \right) + \frac{J_x - J_y}{4} \left(S^+_{j+1} S^+_j + \text{H.c.} \right) \right\}, \tag{9}$$

where $S^\pm = S^x \pm S^y$. Using the Jordan Wigner transformation,

$$\begin{aligned} S^z_j &= (c^\dagger{}_j cj - \frac{1}{2}) \\ \mathbf{S}^+_j &= c^\dagger{}_j e^{i\pi \sum_{l<i} \hat{n}_l}, \end{aligned} \tag{10}$$

we have

$$S^+_{j+1} S^-_j = c^\dagger{}_{j+1} c_j,$$

$$S^+_{j+1}S^+_j \quad = \quad -c^\dagger{}_{j+1}c^\dagger{}_j, \tag{11}$$

so that

$$H = -\sum_j t[c^\dagger{}_{j+1}c_j + \text{H.c}] - \sum_j \Delta[c^\dagger{}_{j+1}c^\dagger{}_j + \text{H.c}] \tag{12}$$

where $t = \frac{J_x+J_y}{4}$, $\Delta = \frac{J_y-J_x}{4}$.
(ii) Transforming to a momentum basis, $c^\dagger{}_j = \frac{1}{N}\sum_q d_q e^{ix_j q}$, the Hamiltonian takes the form

$$H = -\sum_{q>0} 2t\cos(qa)[d^\dagger{}_q d_q + d_{-q}d^\dagger{}_{-q}] - \sum_q \Delta[e^{-iqa}d^\dagger{}_q d^\dagger{}_{-q} + \text{H.c}]. \tag{13}$$

Since $d^\dagger{}_q d^\dagger{}_{-q} = -d^\dagger{}_{-q}d^\dagger{}_q$ is an odd function of q, we can replace $\Delta e^{-iqa} \longrightarrow -2i\Delta\sin qa$, to get

$$H = \sum_{q>0}\epsilon_q[d^\dagger{}_q d_q - d_{-q}d^\dagger{}_{-q}] + \sum_{q>0} i\Delta_q[d^\dagger{}_q d^\dagger{}_{-q} - \text{H.c}]. \tag{14}$$

where $\epsilon_q = -2t\cos(qa)$, $\Delta_q = 2\Delta\sin qa$. Notice how the sum over $q > 0$ is needed so that d_q and d_{-q} are independent. Carrying out the Boguilubov transformation

$$\begin{pmatrix} a_q \\ a^\dagger{}_{-q} \end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}\begin{pmatrix} d_q \\ d^\dagger{}_{-q} \end{pmatrix}, \tag{15}$$

then following the results of the last section, the Hamiltonian takes the form

$$H = \sum_{q>0}\omega_{\vec{q}}[a^\dagger{}_q a_q - a_{-q}a^\dagger{}_{-q}] \tag{16}$$

where

$$\begin{aligned} \omega_q &= \left(\epsilon_q^2 + \Delta_q^2\right)^{\frac{1}{2}} = \frac{1}{2}\left[J_x^2 + J_y^2 + 2J_xJ_y\cos(2qa)\right]^{\frac{1}{2}} \\ \begin{pmatrix} u_q \\ v_q \end{pmatrix} &= \begin{pmatrix} \left[1 + \frac{\epsilon_q}{\omega_q}\right]^{\frac{1}{2}} \\ i\left[1 - \frac{\epsilon_q}{\omega_q}\right]^{\frac{1}{2}} \end{pmatrix} \end{aligned} \tag{17}$$

The spectrum of spin-excitations is shown below. For the case $J_y = J_x$, the excitation spectrum is gapless, corresponding to the continuous rotational symmetry (Goldstone mode). For the case J_y, or $J_x = 0$, the excitation spectrum is flat, as expected for the 1d Ising model.

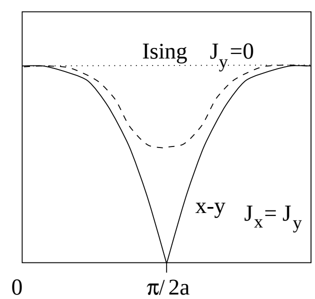

Figure 2: Showing dispersion for x-y, anisotropic x-y and Ising limits of model.