INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Questions 6. Finite Temperature and superfluidity (Due Fri, 12th Dec.)

1. Use the method of complex contour integration to carry out the Matsubara sums in the following:

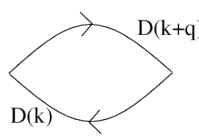
   (i) Derive the density of a spinless Bose Gas at finite temperature from the boson propagator $D(k) \equiv D(\mathbf{k}, \mathbf{i\nu_n}) = [\mathbf{i\nu_n} - \omega_{\mathbf{k}}]^{\mathbf{-1}}$, where $\omega_{\mathbf{k}} = E_{\mathbf{k}} - \mu$ is the energy of a boson, measured relative to the chemical potential.

$$\rho(T) = \frac{N}{V} = V^{-1} \sum_{\mathbf{k}} \langle T b_{\mathbf{k}}(0^-) b^{\dagger}{}_{\mathbf{k}}(0) \rangle = -(\beta V)^{-1} \sum_{i\nu_n, \mathbf{k}} D(k) e^{i\nu_n 0^+}. \tag{1}$$

   How do you need to modify your answer to take account of Bose Einstein condensation?

   (ii) The dynamic charge-susceptibility of a free Bose gas, i.e

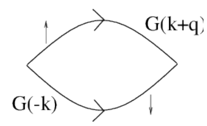
$$\chi_c(q, i\nu_n) =$$



$$= T \sum_{i\nu_r} \int \frac{d^3k}{(2\pi)^3} D(q+k) D(k). \tag{2}$$

   where ν_r is the Bose Matsubara frequency of the internal loop. Please analyticeally extend your final answer to real frequencies ($i\nu_n \rightarrow \nu$).
   "pair-susceptibility" of a spin-1/2 free Fermi gas, i.e.

$$\chi_P(q, i\nu_n) =$$



$$= T \sum_{i\omega_r} \int \frac{d^3k}{(2\pi)^3} G(q+k) G(-k) \tag{3}$$

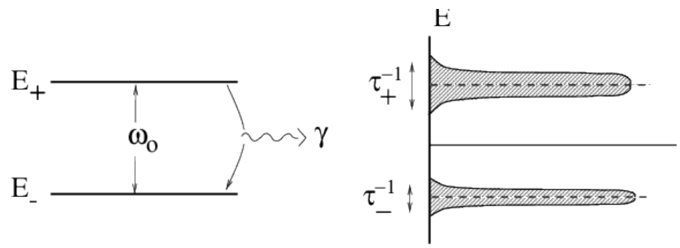
   where $G(k) \equiv G(\mathbf{k}, \mathbf{i\omega_n}) = [\mathbf{i\omega_n} - \epsilon_{\mathbf{k}}]^{\mathbf{-1}}$. (Note the direction of the arrows: why is there no minus sign for the Fermion loop?) Show that the static pair susceptibility, $\chi_P(0)$is given by

$$\chi_P = \int \frac{d^3k}{(2\pi)^3} \frac{\tanh[\beta\epsilon_{\mathbf{k}}/2]}{2\epsilon_{\mathbf{k}}} \tag{4}$$

   Can you see that this quantity diverges at low temperatures? How does it diverge, and why ? (Hint: replace the integration near the Fermi surface by an integration over the density of states, which you may treat as a constant to obtain the divergence)

2. A simple model an atom with two atomic levels coupled to a radiation field is described by the Hamiltonian

$$H = H_o + H_I + H_{\text{photon}}, \tag{5}$$



where

$$H_o = \tilde{E}_- c^\dagger{}_- c_- + \tilde{E}_+ c^\dagger{}_+ c_+ \tag{6}$$

describes the atom, treating it as a *fermion*

$$H_I = V^{-1/2} \sum_{\vec{q}} g(\omega_{\vec{q}}) \left(c^\dagger{}_+ c_- + c^\dagger{}_- c_+ \right) \left[a^\dagger{}_{\vec{q}} + a_{-\vec{q}} \right] \tag{7}$$

describes the coupling to the radiation field (V is the volume of the box enclosing the radiation) and

$$H_{photon} = \sum_{\vec{q}} \omega_{\vec{q}} a^\dagger{}_{\vec{q}} a_{\vec{q}}, \qquad (\omega_q = cq) \tag{8}$$

is the Hamiltonian for the electromagnetic field. The "dipole" matrix element $g(\omega)$ is weak enough to be treated by second order perturbation theory and the polarization of the photon is ignored.

(i) Calculate the self-energy $\Sigma_+(\omega)$ and $\Sigma_-(\omega)$ for an atom in the $+$ and $-$ states.

(ii) Use the self-energy obtained above to calculate the life-times $\tau_\pm$ of the atomic states, i.e.

$$\tau_\pm^{-1} = 2\text{Im}\Sigma_\pm(\tilde{E}_\pm - i\delta). \tag{9}$$

If the gas of atoms is non-degenerate, i.e the Fermi functions are all small compared with unity, $f(E_\pm) \sim 0$ show that

$$\begin{aligned} \tau_+^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o)[1 + n(\omega_o)] \\ \tau_-^{-1} &= 2\pi |g(\omega_o)|^2 F(\omega_o) n(\omega_o), \end{aligned} \tag{10}$$

where $\omega_o = \tilde{E}_+ - \tilde{E}_-$ is the separation of the atomic levels and

$$F(\omega) = \int \frac{d^3q}{(\mathbf{2}\pi)^3} \delta(\omega - \omega_q) = \frac{\omega^2}{2\pi c^3} \tag{11}$$

is the density of state of the photons at energy ω. What do these results have to do with stimulated emission? Do your final results depend on the initial assumption that the atoms were fermions?

(iii) Why is the decay rate of the upper state larger than the decay rate of the lower state by the factor $[1+n(\omega_0)]/n(\omega_0)$? (Hint: appeal to detailed balance)

3. A one dimensional superconductor of *spinless* electrons is described by a p-wave BCS model

$$H = \sum_k \epsilon_k {c^\dagger}_k c_k - \frac{g}{L} \sum_{k,k'} \gamma_k \gamma_{k'} ({c^\dagger}_k {c^\dagger}_{-k})(c_{-k'} c_{k'}) \tag{12}$$

where $\epsilon_k = -2t\cos k - \mu$ is the disperson of the electrons, while $\gamma_k = \sin k$ is the (odd-parity) form factor of the pairing. Consider periodic boundary conditions on a chain with L sites, so that the momenta take the discrete values $k = \frac{2\pi n}{L}$ where n is an integer and $k \in [-\pi, \pi]$.

(i) Why is must γ_k be an odd-parity function of momentum, i.e $\gamma_k = -\gamma_{-k}$?

(ii) Show that the BCS mean-field Hamiltonian for this model takes the form

$$H_{BCS} = \sum_{k\in[0,\pi]} {\psi^\dagger}_k \begin{pmatrix} \epsilon_k & 2\Delta\gamma_k \\ 2\bar{\Delta}\gamma_k & -\epsilon_k \end{pmatrix} \psi_k + L\frac{\bar{\Delta}\Delta}{g} - L\mu$$

where $\bar{\Delta} = \Delta^*$ and

$$\psi_k = \begin{pmatrix} c_k \\ {c^\dagger}_{-k} \end{pmatrix}, \qquad {\psi^\dagger}_k = ({c^\dagger}_k, c_{-k})$$

are Nambu spinors. (Key question: why does the summation run over half the Brillouin zone?)

(iii) What is the mean-field relationship between Δ and the fermion pair density $\langle c_{-k} c_k \rangle$?

(iv) Calculate the Nambu Greens function $\mathcal{G}_{\alpha\beta}(k, i\omega_n) = -\int_0^\beta d\tau \langle T\psi_{k\alpha}(\tau){\psi_{k\beta}}^\dagger(0)\rangle$ using the above mean-field Hamiltonian. How does the anomalous (Gor'kov) propagator

$$F(k, i\omega_n) = -\int_0^\beta \langle T c_k(\tau) c_{-k}(0)\rangle e^{i\omega_n \tau}$$

differ from an s-wave superconductor?

(v) Calculate the quasiparticle excitation spectrum of this superconductor.

(vi) Calculate the free energy as a function of $\Delta = |\Delta|e^{i\phi}$. Plot or sketch your result, illustrating the phase *and* amplitude dependence of the Free energy. One way to do this is to sum the Free energy of the quasiparticles, adding in the term $L|\Delta|^2/g$. At what value of Δ is the free energy minimized?

(vii) Rewrite the mean-field Hamiltonian of 3(b) in real space for the special case where $\Delta = it$ and $\mu = 0$. What happens to the excitation spectrum when the link between the first and last sites is cut?