INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2025

Answers to Questions III. Oct. 20th

1. (a) The eigenvalues are as follows:

$$\begin{aligned} |1_{\mathbf{k}}\rangle &= b^{\dagger}{}_{\mathbf{k}}|0\rangle, & \mathcal{E}_{1_{\mathbf{k}}} &= \epsilon_{\mathbf{k}} \\ |2_{\mathbf{k}}\rangle &= \frac{1}{\sqrt{2!}}(b^{\dagger}{}_{\mathbf{k}})^2|0\rangle, & \mathcal{E}_{2_{\mathbf{k}}} &= 2\epsilon_{\mathbf{k}} \\ |3_{\mathbf{k}}\rangle &= \frac{1}{\sqrt{3!}}(b^{\dagger}{}_{\mathbf{k}})^3|0\rangle, & \mathcal{E}_{3_{\mathbf{k}}} &= 3\epsilon_{\mathbf{k}} + U \end{aligned} \tag{1}$$

Notice that the excitation energies are $\epsilon_{\mathbf{k}} = \mathcal{E}_{1_{\mathbf{k}}} = \mathcal{E}_{2_{\mathbf{k}}} - \mathcal{E}_{1_{\mathbf{k}}}$ and $\epsilon_{\mathbf{k}} + U = \mathcal{E}_{3_{\mathbf{k}}} - \mathcal{E}_{2_{\mathbf{k}}}$.

(b) If we can ignore occupancies higher than three, then the partition function is

$$Z = \prod_{\mathbf{k}} (1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)})$$

so that the free energy is

$$F = \sum_{\mathbf{k}} F_{\mathbf{k}} = -k_B T \sum_{\mathbf{k}} \ln\left[1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)}\right]$$

(c) The occupancy of the $\mathbf{k}$ state is

$$n_{\mathbf{k}} = \langle n_{\mathbf{k}}\rangle = -\frac{\partial F_{\mathbf{k}}}{\partial \mu} = \frac{e^{-\beta\epsilon_{\mathbf{k}}} + 2e^{-2\beta\epsilon_{\mathbf{k}}} + 3e^{-\beta(3\epsilon_{\mathbf{k}}+U)}}{1 + e^{-\beta\epsilon_{\mathbf{k}}} + e^{-2\beta\epsilon_{\mathbf{k}}} + e^{-\beta(3\epsilon_{\mathbf{k}}+U)}}$$

(d) Let us plot $n_{\mathbf{k}}$ at low temperatures. There are three regions to consider:

- $\epsilon_{\mathbf{k}} > 0$, $n_{\mathbf{k}} = 0$.
- $\epsilon_{\mathbf{k}} < 0$, but $\epsilon_{\mathbf{k}} + U > 0$, $n_{\mathbf{k}} = 2$.
- $\epsilon_{\mathbf{k}} + U < 0$ and $n_{\mathbf{k}} = 3$. so that there are two "Fermi surfaces" (see Fig. 1).

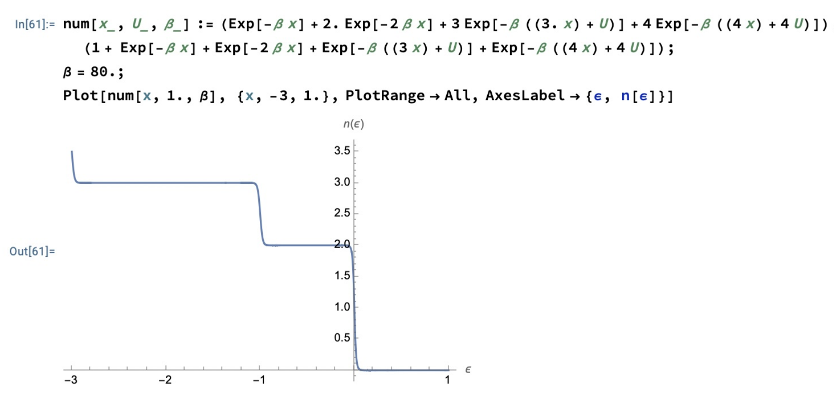

Figure 1: The occupancy versus $\epsilon_{\mathbf{k}}$, showing two Fermi surfaces.

2. (a) We may estimate the Bose Einstein transition temperature from

$$T_{BE} = \frac{3.31}{k_B}\left(\frac{\hbar^2 n^{2/3}}{m}\right) = \frac{3.31}{1.38\times 10^{-23}}\left(\frac{\hbar^2(10^{21}m^{-3})^{2/3}}{23m_p}\right) \approx 6.9\mu K.$$

These tiny temperatures are attained by "evaporative cooling" . Sodium atoms are held in a "magneto-optic" trap. Radio waves are used to "evaporate" the most energetic atoms in the trap, leaving behind the cold ones.

(b) In Helium-4, we may estimate the Bose Einstein transition temperature as

$$T_{BE} = \frac{3.31}{k_B}\left(\frac{\hbar^2 n^{2/3}}{m_{He}}\right) = \frac{3.31}{1.38\times 10^{-23}}\left(\frac{\hbar^2((122/(4m_p)))^{2/3}}{4m_p}\right) \approx 2.76K.$$

The actual condensation temperature is 2.21K. The difference in condensation temperatures is due to the repulsive interaction between atoms.

3. (a) If the interaction has the form

$$V(r) = \begin{cases} U, & (r < R), \\ 0, & (r > R), \end{cases} \tag{2}$$

then in second-quantized form, the interaction Hamiltonian is

$$V = \frac{U}{2}\sum_{\sigma,\sigma'}\int d^3x \int_{|\vec{x}'-\vec{x}|<R} d^3x' [\psi^\dagger{}_\sigma(x)\psi^\dagger{}_{\sigma'}(x')\psi_{\sigma'}(x')\psi_\sigma(x)]. \tag{3}$$

(ii) Inverting the Fourier transform, we have $c_{\vec{k}\sigma} = \int d^3x\psi_\sigma(\vec{x})e^{-i\vec{k}\cdot\vec{x}}$, so that

$$\begin{aligned}[c_{\vec{k}\sigma},\ c^\dagger{}_{\vec{k}'\sigma'}]_\pm &= \int d^3x d^3x' [\psi_\sigma(x),\ \psi^\dagger{}_{\sigma'}(x')]_\pm e^{-i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')} \\ &= \delta_{\sigma\ \sigma'}\int d^3x e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} \\ &= \delta_{\sigma\ \sigma'}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'). \end{aligned} \tag{4}$$

(iii) In momentum space, we may write

$$V = \frac{1}{2}\int \frac{d^3k d^3k' d^3q}{(2\pi)^9} V(q)\left[c^\dagger{}_{\vec{k}+\vec{q}\sigma}c^\dagger{}_{\vec{k}'-\vec{q}\sigma'}c_{\vec{k}'\sigma'}c^\dagger{}_{\vec{k}\sigma}\right], \tag{5}$$

where

$$V(\vec{q}) = \int d^3x V(\vec{x})e^{i\vec{q}\cdot\vec{x}} = \frac{4\pi U}{q}\int_0^R dr r\sin(qr) = \left(\frac{4\pi R^3 U}{3}\right)F(qR) \tag{6}$$

and

$$F(x) = \frac{3}{x^2}\left[\frac{\sin x}{x} - \cos x\right]. \tag{7}$$

The form of the interaction in momentum space is sketched above. The hard core in real space is manifested as a long-range oscillatory component in momentum space.

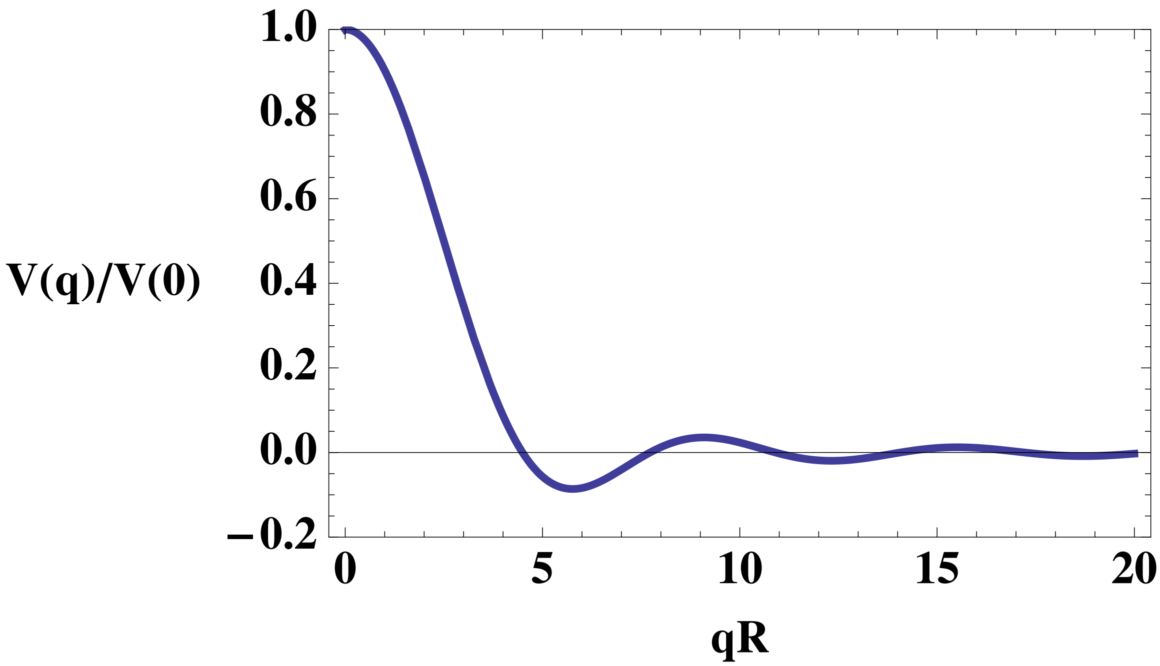

Figure 2: Fourier transformed potential $V(q)$ for "hard sphere" potential.