682A. Exercises 3. Due April 22nd

1. Generalize the Cooper pair calculation to higher angular momenta. Consider an interaction that has an attractive component in a higher angular momentum channel, such as

$$N(0)V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -g_l(2l+1)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'), & (|\epsilon_{\mathbf{k}}|, |\epsilon'_{\mathbf{k}}| < \omega_0), \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

   where you may assume l is even.

   (a) By decomposing the Legendre Polynomial in terms of spherical harmonics, $(2l+1)P_l(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = 4\pi \sum_m Y_{lm}(\mathbf{k})Y^*_{lm}(\hat{\mathbf{k}})$, show that this interaction gives rise to bound Cooper pairs with a finite angular momentum, given by

   $$|\psi_P\rangle = \sum_{\mathbf{k}} \phi_{\mathbf{k}m} Y_{lm}(\hat{\mathbf{k}}) c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow}|0\rangle$$

   with a bound-state energy given by

   $$E = -2\omega_0 \exp\left[-\frac{2}{g_l N(0)}\right]$$

   (b) A general interaction will have several harmonics:

   $$V_{\mathbf{k},\mathbf{k}'} = \frac{1}{V}\sum_l g_l(2l+1)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'),$$

   not all of them attractive. In which channel (s) will the pairs tend to condense?

   (c) Why can't you use this derivation for the case when l is odd?

2. Explicit calculation of the Free energy.

   (a) Assuming that the Debye frequency is a small fraction of the band-width, show that the difference between the superconducting and normal state Free energy can be written as the integral

   $$\mathcal{F}_S - \mathcal{F}_N = -2TN(0)\int_{-\omega_D}^{\omega_D} d\epsilon \ln\left[\frac{\cosh\left(\frac{\sqrt{\epsilon^2+|\Delta|^2}}{2T}\right)}{\cosh\left(\frac{\epsilon}{2T}\right)}\right] + V\frac{|\Delta|^2}{g_0}.$$

   Why is this free energy invariant under changes in the phase of the gap parameter $\Delta \to \Delta e^{i\phi}$?

(b) By differentiating the above expression with respect to Δ, confirm the zero temperature gap equation,

$$\frac{V}{gN(0)} = \int_0^{\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta_0^2}},$$

where $\Delta_0 = \Delta(T = 0) =$ is the zero temperature gap and use this result to eliminate g_0, to show that the free energy can be written

$$\mathcal{F}_S - \mathcal{F}_N = N(0)\Delta_0^2 \, \Phi\left[\frac{\Delta}{\Delta_0}, \frac{T}{\Delta_0}\right]$$

where the dimensionless function

$$\Phi(\delta, t) = \int_0^{\infty} dx \left\{ -4t \ln\left[\frac{\cosh\left(\frac{\sqrt{x^2+\delta^2}}{2t}\right)}{\cosh\left(\frac{x}{2t}\right)}\right] + \frac{\delta^2}{\sqrt{x^2 + 1}} \right\}.$$

Here, the limit of integration have been moved to infinity. Why can we do this without loss of accuracy?

(c) Use Mathematica or Maple to plot the Free energy obtained from the above result, confirming that the minimum is at $\Delta/\Delta_0 = 1$ and the transition occurs at $T_c = 2\Delta_0/3.53$.

3. The standard two-component Nambu spinor approach does not allow a rotationally invariant treatment of the electron spin and the Zeeman coupling of fermions to a magnetic field. This drawback can be overcome by switching to a four-component "Balian Werthammer" spinor, denoted by

$$\psi_{\mathbf{k}} = \begin{pmatrix} c^{\dagger}{}_{\mathbf{k}} \\ -i\sigma_2 (c^{\dagger}{}_{\mathbf{k}})^T \end{pmatrix} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ -c_{\mathbf{k}\downarrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \\ c^{\dagger}{}_{-\mathbf{k}\uparrow} \end{pmatrix}. \tag{2}$$

(a) Show using this notation that the total electron spin can be written

$$\vec{S} = \frac{1}{4} \sum_{\mathbf{k}} \psi^{\dagger}{}_{\mathbf{k}} \vec{\sigma}_{(4)} \psi_{\mathbf{k}} \tag{3}$$

where

$$\vec{\sigma}_4 = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{4}$$

is the four component Pauli matrix. (You may find it useful to use the relationship $\vec{\sigma}^T = i\sigma_2 \vec{\sigma} i\sigma_2$). In practical usage, the subscript "4" is normally dropped.

(b) Show that in a Zeeman field, the BCS Hamiltonian

$$H_{MFT} = \sum_{\mathbf{k}\sigma} c^{\dagger}{}_{\mathbf{k}\alpha}[\epsilon_{\mathbf{k}}\delta_{\alpha\beta} - \vec{\sigma}_{\alpha\beta} \cdot \vec{B}]c_{\mathbf{k}\beta} + \sum_{\mathbf{k}} \left[\bar{\Delta}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} + c^{\dagger}{}_{\mathbf{k}\uparrow}c^{\dagger}{}_{-\mathbf{k}\downarrow}\Delta\right] + \frac{V}{g_0}\bar{\Delta}\Delta \tag{5}$$

can be re-written using Balian Werthammer spinors in the compact form

$$H_{MFT} = \frac{1}{2}\sum_{\mathbf{k}} \psi^{\dagger}{}_{\mathbf{k}} \left[\underline{h}_{\mathbf{k}} - \vec{\sigma}_4 \cdot \vec{B}\right] \psi_{\mathbf{k}} + \frac{V}{g_0}\bar{\Delta}\Delta \tag{6}$$

where $\underline{h}_{\mathbf{k}} = \epsilon_{\mathbf{k}}\tau_1 + \Delta_1\tau_1 + \Delta_2\tau_2$ as before, but the $\vec{\tau}$ now refer to the four-dimensional Nambu matrices

$$\vec{\tau} = \left(\begin{bmatrix} 0 & \underline{1} \\ \underline{1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i\underline{1} \\ i\underline{1} & 0 \end{bmatrix}, \begin{bmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{bmatrix}\right). \tag{7}$$

(c) Show that the quasiparticle energies in a field are given by $\pm E_{\mathbf{k}} - \sigma B$.

(d) Suppose that the magnetic field only coupled to the spin of the electron, not to its current. What would the "Pauli-limited" upper critical field of the superconductor be? (i.e the field at which the gap would go to zero)

4. The Nambu Green's function for He-3B is given by

$$\mathcal{G}(\mathbf{k}, i\omega_n) = [i\omega_n - \epsilon_{\mathbf{k}}\tau_3 - \Delta\hat{\mathbf{k}} \cdot \vec{\sigma}\tau_1]^{-1} \tag{8}$$

(a) Explain how you would calculate the superfluid stiffness for this neutral superfluid? In a charged superconductor, one can calculate such a stiffness by introducing an external vector potential. However, At first sight, there are no physical gauge fields that couple to the current in a neutral superfluid. Is this a problem for your calculation?

(b) Carry out your calculation and write down an equation that generalizes the superfluid stiffness of a BCS singlet state to the B-phase of He-3B.

(c) In superfluid Helium-3, if you twist the phase of the order parameter, you induce a mass superflow. What happens if you twist the direction (in spin space) of the superfluid order parameter?