682A. Exercises 1. Due Feb 16th.

1. Grassman algebra and calculus. Calculate the following expressions to gain familiarity with Grassmanian (fermionic) numbers Functions are to be calculated using Taylor expansions in terms of the Grassmanian components, truncating whenever you have a Grassman squared, e.g. $e^{\alpha} = 1 + \alpha$ Assume that Greek-lettered variables are Grassman numbers.

   (a) $(1+\alpha)^{-1}$. (Why can't you calculate the inverse of α, α^{-1} ?)

   (b) $\sqrt{(1+\bar{\alpha}\alpha)}$

   (c) $\cos[\bar{\alpha}\alpha + \bar{\beta}\beta]$

   (d) $\exp\begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}$.

   (e) Suppose $f(\bar{\alpha}, \alpha) = f + \bar{\beta}\alpha - \bar{\alpha}\beta + g\bar{\alpha}\alpha$ is a Grassmanian function of two variables. Calculate: (i) $\dfrac{\partial f(\alpha,\alpha)}{\partial \alpha}$, (ii) $\dfrac{\partial^2 f(\alpha,\alpha)}{\partial\bar{\alpha}\partial\alpha}$, (iii) $\displaystyle\int d\bar{\alpha}d\alpha f(\bar{\alpha},\alpha)$.

   (f) $\displaystyle\int d\bar{\alpha}_1 d\alpha_1 d\bar{\alpha}_2 d\alpha_2 \exp\left[(\bar{\alpha}_1, \bar{\alpha}_2)\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\right]$.

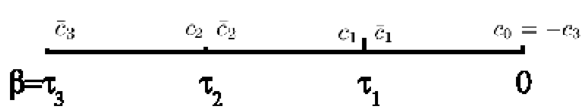
2. (a) Suppose $H = \epsilon c^{\dagger} c$ is a fermionic Hamiltonian ($\{c, c^{\dagger}\} = 1$). Consider the approximation to the partition function obtained by dividing up the period $\tau \in [0, \beta]$ into N equal time-slices,

$$Z_N = \mathrm{Tr}[(e^{-\Delta\tau H})^N] \tag{1}$$

where $\Delta\tau = \beta/N$. By using coherent states $|c\rangle = e^{\hat{c}^{\dagger} c}|0\rangle$, approximating the matrix element from time τ_j to time τ_{j+1}, where $\tau_j = j\Delta\tau$ by

$$\langle \bar{c}_{j+1}| e^{-\Delta\tau H} |c_j\rangle = e^{\alpha \bar{c}_{j+1} c_j} + O(\Delta\tau^2) \tag{2}$$

where $\alpha = (1 - \Delta\tau\epsilon)$,



(3)

and making use of the Trace formula

$$\mathrm{Tr}A = \int d\bar{c}_3 dc_0 \langle -\bar{c}_3 | A | c_0\rangle e^{-\bar{c}_3 c_0}, \tag{4}$$

show that Z_3 can be written as a "toy functional integral",

$$Z_3 = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{ -(\bar{c}_3, \bar{c}_2, \bar{c}_1)\begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{bmatrix}\begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix} \right\}. \tag{5}$$

(b) Evaluate Z_3.

(c) Generalize the result to N time slices and obtain an expression for Z_N. What is the limiting value of your result as $N \to \infty$?

3. Using path integrals, calculate the partition function for a single Zeeman-split electronic level (field in the z-direction) with energy $-\sigma_z B$described by the action

$$S = \int d\tau \bar{f}_\sigma \Big(\partial_\tau - \sigma B \Big) f_\sigma,$$

where B is the applied field. Why is your answer not the same as the partition function of a spin $S = 1/2$ in a magnetic field? Is there a way you could alter the action to improve the answer?

4. Consider the operator $\hat{A} = A_0 + A_1 \hat{c}^\dagger \hat{c}$ where $\hat{c}^\dagger$ is a fermion creation operator. Clearly $Tr[\hat{A}] = 2A_0 + A_1$. Confirm that you obtain the same result using the trace formula:

$$\mathrm{Tr}[\hat{A}] = \int d\bar{c} dc \langle -c|\hat{A}|c\rangle e^{-\bar{c}c} \tag{6}$$

(Note that $\langle -c|\hat{A}|c\rangle = A[-\bar{c}, c]e^{-\bar{c}c}$.)

5. More challenging. Suppose

$$\mathcal{M} = e^{\frac{1}{2}\sum_{i,j} A_{ij} c^\dagger{}_i c^\dagger{}_j}$$

where A_{ij} is an $N \times N$ antisymmetric matrix, and the $c^\dagger{}_j$ are a set of N canonical Fermi creation operators. Using coherent states, calculate

$$\mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger]$$

where the trace is over the 2^N dimensional Hilbert space of fermions.

(Hints: notice that $\mathcal{M}\mathcal{M}^\dagger$ is already normal ordered, so that you can write its matrix element in coherent states. Next, using the trace formula, you can rewrite this in terms of a simple Grassman integral that can be evaluated. You may find it useful to use or derive the Coherent state representation for the overlap between two fermion states,

$$\langle l, m|s, r\rangle = \int \prod_{i=1,N} d\bar{c}_i dc_i c_l c_m \bar{c}_r \bar{c}_s e^{-\sum_i \bar{c}_i c_i} = \delta_{ls}\delta_{mr} - \delta_{lr}\delta_{sm} \tag{7}$$

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