Many Bosons

$$|b\rangle = \exp\left(\sum_\lambda b_\lambda^\dagger b_\lambda\right),$$

$$1 = \int \prod \frac{d\bar{b}_\lambda db_\lambda}{2\pi i} e^{-\bar{b}_\lambda b_\lambda} |b\rangle\langle \bar{b}|,$$

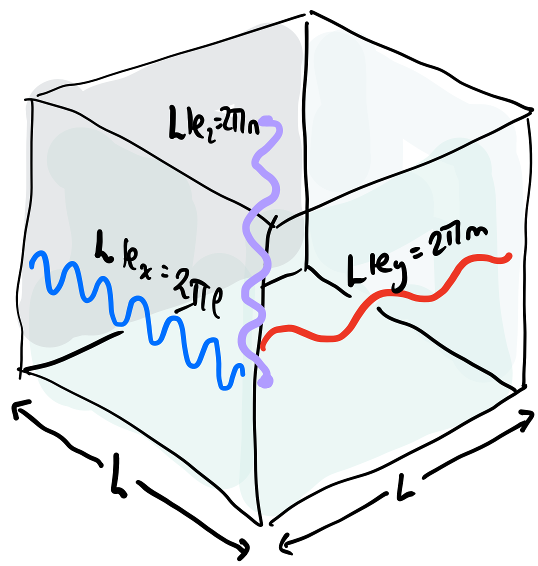
$$D[\bar{b},b] = \prod_\lambda D[\bar{b}_\lambda, b_\lambda].$$

$$S = \sum_\lambda \int_0^\beta d\tau\ \bar{b}_\lambda(\partial_\tau + \omega_\lambda) b_\lambda \qquad \text{Free bosons}$$

$$\boxed{S = \int_0^\beta d\tau \left(\sum \bar{b}_\lambda \partial_\tau b_\lambda + H[\bar{b},b]\right) \qquad \text{More Generally}}$$

e.g.
$$\begin{cases} \omega_{\vec{k}} = E_k - \mu \\ E_k = \vec{k}^2/2m \end{cases}$$

$$\vec{k} = \frac{2\pi}{L}(\ell, m, n)$$



$$S = \sum_{\vec{k}} \int_0^\beta d\tau\ \bar{b}_k(\partial_\tau + \omega_{\vec{k}}) b_k$$

FOURIER SPACE.

$$b_k(\tau) = \frac{1}{\sqrt{\beta}} \sum b_{kn} e^{-i\nu_n \tau} \implies \nu_n \beta = 2\pi \times \text{integer}$$

$$\boxed{\nu_n = \frac{2\pi n}{\beta} = 2\pi k_B T n}$$

MATSUBARA FREQUENCY (BOSONS).

$$b_k(\tau) = b_k(\beta) \quad \text{Periodic B.C's}$$

$$S = \frac{1}{\beta} \sum_k \sum_{nm} \int \bar{b}_{kn} e^{i\nu_n \tau} (\partial_\tau + \omega_k) b_{km} e^{-i\nu_m \tau} d\tau$$

$$= \sum_k \sum_{n,m} \bar{b}_{kn} (-i\nu_m + \omega_k) b_{km} \underbrace{\frac{1}{\beta}\int_0^\beta d\tau\, e^{i(\nu_n - \nu_m)\tau}}_{= \delta_{nm}}$$

$$= \sum_{kn} \bar{b}_{kn} (-i\nu_n + \omega_k) b_{kn}$$

$$b_k(\tau) = U_{\tau n} b_{kn} \qquad U_{\tau n} = \frac{1}{\sqrt{\beta}} e^{-i\nu_n \tau} \equiv U_{\tau n}, \quad U^\dagger_{n\tau} = \frac{1}{\sqrt{\beta}} e^{i\nu_n \tau}.$$

$$U^\dagger U = \int d\tau\, U^\dagger_{m\tau} U_{\tau n} = \delta_{nm} \equiv 1$$

$$U U^\dagger = \sum_n U_{\tau n} U_{n\tau'} = \sum_m \delta(\tau - \tau' - m\beta) \equiv 1$$

$$\mathcal{D}[\bar{b}, b] = \prod \frac{d\bar{b}_k(\tau)\, db_k(\tau)}{2\pi i} = \prod \frac{d\bar{b}_{kn}\, db_{kn}}{2\pi i} \frac{\delta[\bar{b}_k(\tau), b_k(\tau)]}{\delta[\bar{b}_{kn}, b_{kn}]}$$

$$= \prod \frac{d\bar{b}_{kn}\, db_{kn}}{2\pi i} \underbrace{\left\| \begin{matrix} U^\dagger & \\ & U \end{matrix} \right\|}_{\det[U^\dagger U]} = \prod \frac{d\bar{b}_{kn}\, db_{kn}}{2\pi i}$$

$$\boxed{\text{Unitary transformation of bosons} \implies \text{measure unchanged.}}$$

$$Z = e^{-\beta F} = \int \prod \frac{d\bar{b}_{kn} db_{kn}}{2\pi i} \exp\left[-\sum \bar{b}_{kn} b_{kn}(-i\nu_n + \omega_k)\right] = \frac{1}{\prod_{kn}(-i\nu_n + \omega_k)}$$

$$\Rightarrow \quad F = k_B T \sum_{\vec{k},n} \ln(\omega_k - i\nu_n)\, \underset{\uparrow}{e^{i\nu_n 0^+}}$$

convergence factor.
added for later.

Real Space

$$b_k(\tau) = \frac{1}{\sqrt{V}} \int d^3x\, \psi(x)\, e^{-i\vec{k}\cdot\vec{x}}, \qquad \psi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\, b_{\vec{k}}$$

$$\langle k| = \langle k|x\rangle\langle x| \qquad\qquad \langle x| = \langle x|k\rangle\langle k|$$

$$\left(-\frac{\nabla^2}{2m} - \mu\right)\psi(x) = \frac{1}{\sqrt{V}} \sum e^{i\vec{k}\cdot\vec{x}} \underbrace{\left(\frac{k^2}{2m} - \mu\right)}_{\omega_k} b_{\vec{k}}$$

$$\int_0^\beta d\tau \int d^3x\, \bar{\psi}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi = \sum_{k,k'} \int_0^\beta d\tau\, \bar{b}_{\vec{k}} (\partial_\tau - \omega_{k'}) b_{k'} \underbrace{\frac{1}{V}\int d^3x\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{\delta_{\vec{k}\vec{k}'}}$$

$$= \sum_k \int d\tau\, \bar{b}_k (\partial_\tau - \omega_k) b_k.$$

DIFFERENT REPS.

$$S = \sum_k \int d\tau \; \bar{b}_k \left(\partial_\tau + \omega_k\right) b_k$$

$$\longrightarrow \sum_{kn} \bar{b}_{kn} \left(-i\nu_n + \omega_k\right) b_{kn}$$

FOURIER

$$\longrightarrow \int d\tau \, d^3x \;\; \bar{\psi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu \right) \psi$$

REAL SPACE.

$$-\underline{g}^{-1} = \left(\partial_\tau + \omega_k\right) \equiv \left(-i\nu + \omega_k\right) \equiv \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)$$

THREE REPS OF SAME MATRIX.

$$\underline{g} = \frac{1}{\left(i\nu_n - \omega_k\right)} \delta_{kk'} \delta_{nn'}$$ is called the "propagator"

DIFFERENT REPS:

$$F = T \sum_{kn} \ln\left[\omega_k - i\nu_n\right] = T \,\mathrm{Tr} \ln \Big[\overbrace{(\omega_k - i\nu_n)\delta_{nn'}\delta_{kk'}}^{-g^{-1}} \Big]$$

$$= T \,\mathrm{Tr} \ln \left[-g^{-1}\right]$$

$$= T \ln \prod_{kn} \left[\omega_k - i\nu_n\right]$$

$$= T \ln \det \left[-g^{-1}\right]$$

$$F = T \,\mathrm{Tr} \ln\left[-g^{-1}\right] = T \ln \det \left[-g^{-1}\right]$$

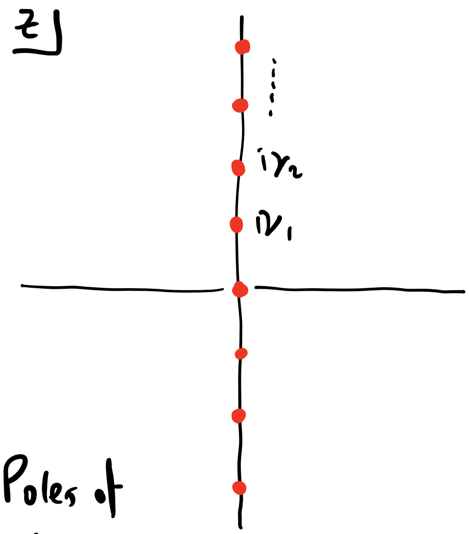
$$Z = \int \mathcal{D}[\bar{b}, b] \exp\Big[- \bar{b} \overbrace{\left(-g^{-1}\right)}^{M} b\Big] = \frac{1}{\det\left[-g^{-1}\right]} \equiv \frac{1}{\prod(\omega_k - i\nu_n)}$$

Matsubara Sums

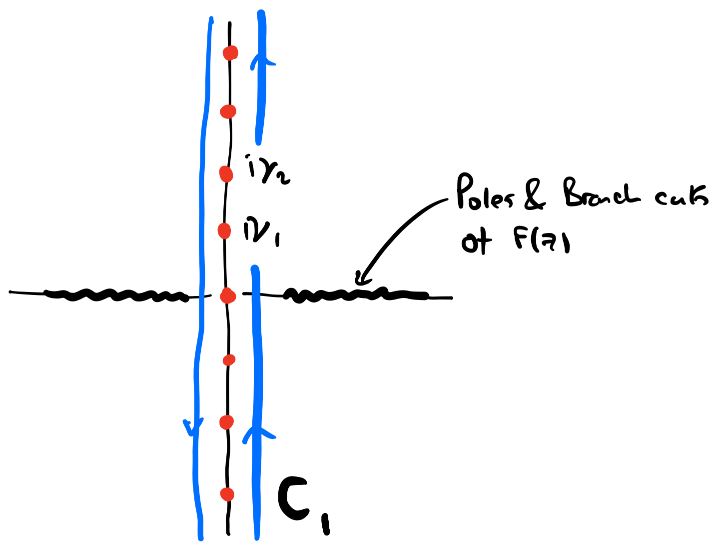
$z = i\nu_n$ are the poles of the Bose function $n(z) = \dfrac{1}{e^{\beta z}-1}$.

$$z = i\nu_n + \delta \Rightarrow n(i\nu_n+\delta) = \frac{1}{e^{i\beta\nu_n + \delta\beta}-1} = \frac{1}{e^{\delta\beta}-1} \approx \frac{k_B T}{\delta}$$

$n(z) \sim \dfrac{k_B T}{(z - i\nu_n)}$ when z is close to $i\nu_n$

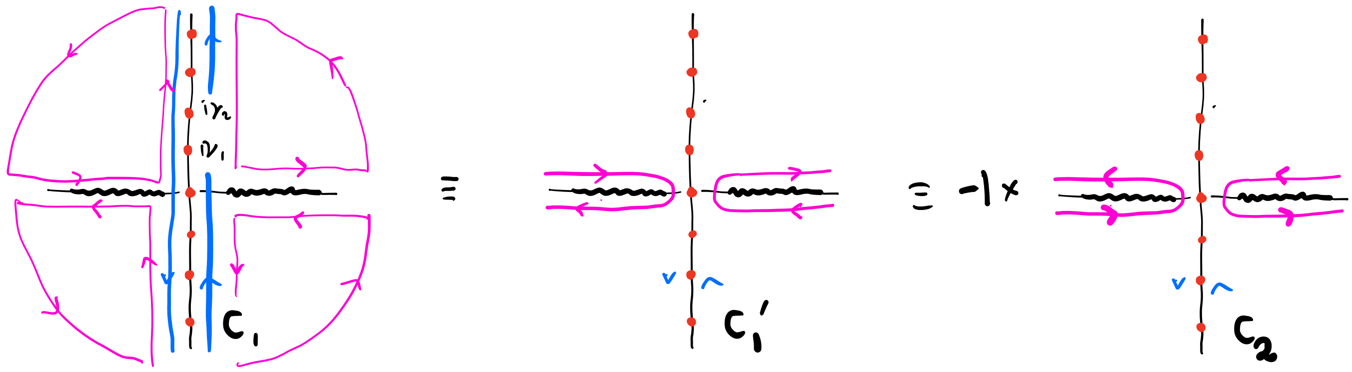


Poles of $n(z)$



$$T\sum_n F(i\nu_n) = \oint_{C_1} \frac{dz}{2\pi i}\, n(z) F(z) = -\oint_{C_2} \frac{dz}{2\pi i}\, n(z) F(z)$$

C_2 Around poles + Branch cuts of F(z)



$$\boxed{T \sum P(i\nu_n)\, e^{i\nu_n 0^+} = -\oint_{C_2} n(z) P(z)\, e^{z 0^+}}$$

required if $|F(z)|$ does not decay faster than $\frac{1}{|z|}$.

e.g.

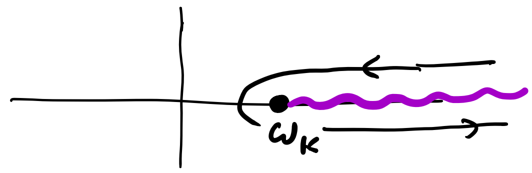
$$\langle \hat{N} \rangle = \Big\langle \sum_{\vec{k}} \hat{b}_k^{+} \hat{b}_k \Big\rangle = -\frac{\partial I}{\partial \mu} = -\frac{\partial}{\partial \mu} \Big[T \sum_{kn} \ln(E_k - \mu - i\nu_n)\, e^{i\nu_n 0^+} \Big]$$

$$= -T \sum \underbrace{\frac{e^{i\nu_n 0^+}}{(i\nu_n - \omega_k)}}_{P = \sum \frac{1}{(\omega_k - i\nu_n)}}$$

$$\langle \hat{N} \rangle = \sum_k \oint_C \frac{dz}{2\pi i}\, n(z) \frac{1}{z - \omega_k} = \sum_k n(\omega_k)$$

$$F = T \sum_{\vec{k}, n} \ln(\omega_k - i\nu_n)\, e^{i\nu_n 0^+}$$

$$P(z) = \sum_k \ln(\omega_k - z)\, e^{z 0^+}$$



$$\therefore \quad F = -\sum_k \oint_C \frac{dz}{2\pi i}\, n(z) \ln(\omega_k - z)\, e^{z 0^+}$$

$$= -\sum_k \int \frac{d\omega}{2\pi i}\, n(\omega) \Big[\overbrace{\ln(\omega_k - (\omega - i\delta))}^{\ln|\omega_k - \omega| + i\pi\theta(\omega - \omega_k)} - \underbrace{\ln(\omega_k - (\omega + i\delta))}_{\ln|\omega_k - \omega| - i\pi\theta(\omega - \omega_k)} \Big]$$

$$= -\sum_k \int_{\omega_k}^{\infty} d\omega\, n(\omega) \qquad = -\sum_k \Big[T \ln(1 - e^{-\beta\omega}) \Big]_{\omega_k}^{\infty}$$

$$\frac{\partial}{\partial\omega} \ln(1 - e^{-\beta\omega}) = \beta \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} = \beta n(\omega)$$

$$\boxed{F = \sum T \ln(1 - e^{-\beta\omega_k})}$$

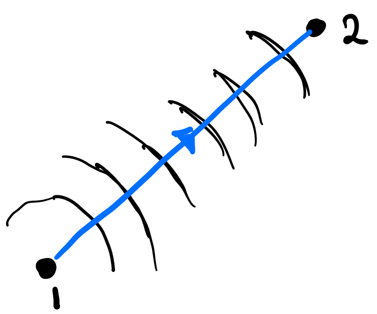
Greens Functions

$$-\langle T\, b_H(2)\, b^\dagger_H(1)\rangle = \mathcal{G}(1,2)$$ Greens Function — FEYNMAN PROPAGATOR.

The Feynman, or time-ordered propagator describes the amplitude for a particle to propagate from $1 \rightarrow 2$. For Fermions the "backwards" time part describes the propagaton of holes or positrons.

$$\mathcal{G}(1,2) = \begin{cases} -\langle \hat{b}_H(1)\, \hat{b}^\dagger_H(2)\rangle & \tau_1 > \tau_2 \\ -\zeta\langle \hat{b}^\dagger_H(2)\, \hat{b}_H(1)\rangle & \tau_1 < \tau_2 \end{cases}$$

$$\zeta = \begin{cases} 1 & \text{Bosons} \\ -1 & \text{Fermions} \end{cases}$$



dummy variable

Here $b_H(\tau) = e^{H\tau} b_s(\tau)\, e^{-H\tau}$ is the Heisenberg operator. We include the dummy label on the Schrödinger operator $b_s(\tau)$

We can write

$$\langle T\, \hat{b}(2)\, b^{+}(1)\rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta \hat{H}}\, T\left(e^{H\tau_2} b_s(2) e^{-H\tau_2}\right)\left(e^{H\tau_1} b_s^{+}(1) e^{-H\tau_1}\right)\right]$$

$$= \frac{1}{Z} \mathrm{Tr}\left[T\; U(\beta-\tau_2)\, b_s(\tau_2)\, U(\tau_2-\tau_1)\, b_s^{+}(\tau_1)\, U(\tau_1\;)\right]$$

$$\left(U(\tau) = e^{-\tau \hat{H}}\right)$$

Note! Ordering important!

$$= \frac{1}{Z} \mathrm{Tr}\left[T\; U(\beta)\, b_s(\tau_1)\, b_s^{+}(\tau_2)\right]$$

$$-\mathcal{G}(2,1) = \frac{1}{Z} \lim_{N\to\infty} \mathrm{Tr}\left[e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}\, \hat{b}_s(\tau_1)\, e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}\, b_s(\tau_2)\, e^{-\Delta\tau H} \cdots e^{-\Delta\tau H}\right]$$

$$\hat{b}_s(\tau_r) = \int \frac{d\bar{b}_r\, db_r}{2\pi i}\, e^{-\bar{b}_r b_r}\, |b_r\rangle\, b_r\, \langle \bar{b}_r|$$

$$b_s^{+}(\tau_s) = \int \frac{d\bar{b}_s\, db_s}{2\pi i}\, e^{-\bar{b}_s b_s}\, |b_s\rangle\, b_s\, \langle \bar{b}_s|$$

$$\therefore\quad \boxed{-\mathcal{G}(2,1) = \underbrace{\langle T\, b_H(2)\, b_H^{+}(1)\rangle}_{\text{TIME ORDERED PROPAGATOR}} = \underbrace{\frac{\int \mathcal{D}[\bar{b},b]\, e^{-S}\, b(2)\,\bar{b}(1)}{\int \mathcal{D}[\bar{b},b]\, e^{-S}}}_{\text{PATH INTEGRAL: TIME ORDERING IS IMPLICIT IN THE INTEGRAL.}}}$$

$$\sum_{c,\bar c} |c\rangle\langle\bar c| = 1 \qquad \langle\bar c|A|c\rangle = e^{\bar c c} A[\bar c, c]$$

$$\mathrm{Tr}\,A = \sum \langle m|A|n\rangle \delta_{nm}$$

$$= \int d\bar c\, dc \sum_{nm} \langle m|A|n\rangle\ e^{-\bar c c}\ \overbrace{c^n\ \bar c^m}^{\langle n|c\rangle\ \langle \bar c|m\rangle}$$

$$= \int d\bar c\, dc\ \langle m|A|n\rangle\langle n|c\rangle\ \langle\bar c|m\rangle\ e^{-\bar c c}$$

$$= \int d\bar c\, dc\ \langle -\bar c|m\rangle\langle m|A|n\rangle\langle n|c\rangle\ e^{-\bar c c}$$

$$\bar c' = -c \qquad \int d\bar c\ dc\ \langle -\bar c|A|c\rangle\, e^{-\bar c c} = -\int d\bar c'\, dc\ e^{\bar c' c}\langle \bar c'|A|c\rangle$$

$$Z_N = -\int d\bar c_N\, dc_0\ e^{\bar c_N c_0} \prod_{j=1}^{N-1} d\bar c_j\, dc_j\, e^{-\bar c_j c_j} \prod_{j=1}^{N} \langle \bar c_j|e^{-\Delta\tau H}|c_{j-1}\rangle$$

$$\bar c_N = -\bar c_0 \qquad c_N = -c_0$$

$$Z_N = \int d\bar c_N\, dc_N\ e^{-\bar c_N c_N} \langle \bar c_N|\dots\ |c_0\rangle$$

$$= \int \prod d\bar c_j\, dc_j\ e^{-\sum\left[\bar c_j \frac{(c_j - c_{j-1})}{\Delta\tau}\Delta\tau + \Delta\tau H[\bar c_j, c_{j-1}]\right]}$$

$$\int \prod d\bar c\, dc\ \exp\left[-\int_0^\beta d\tau\ \bar c(\partial_\tau + \epsilon)c\right]$$

$$c_j = \frac{1}{\sqrt\beta}\sum c_n e^{-i\omega_n \tau_j}$$

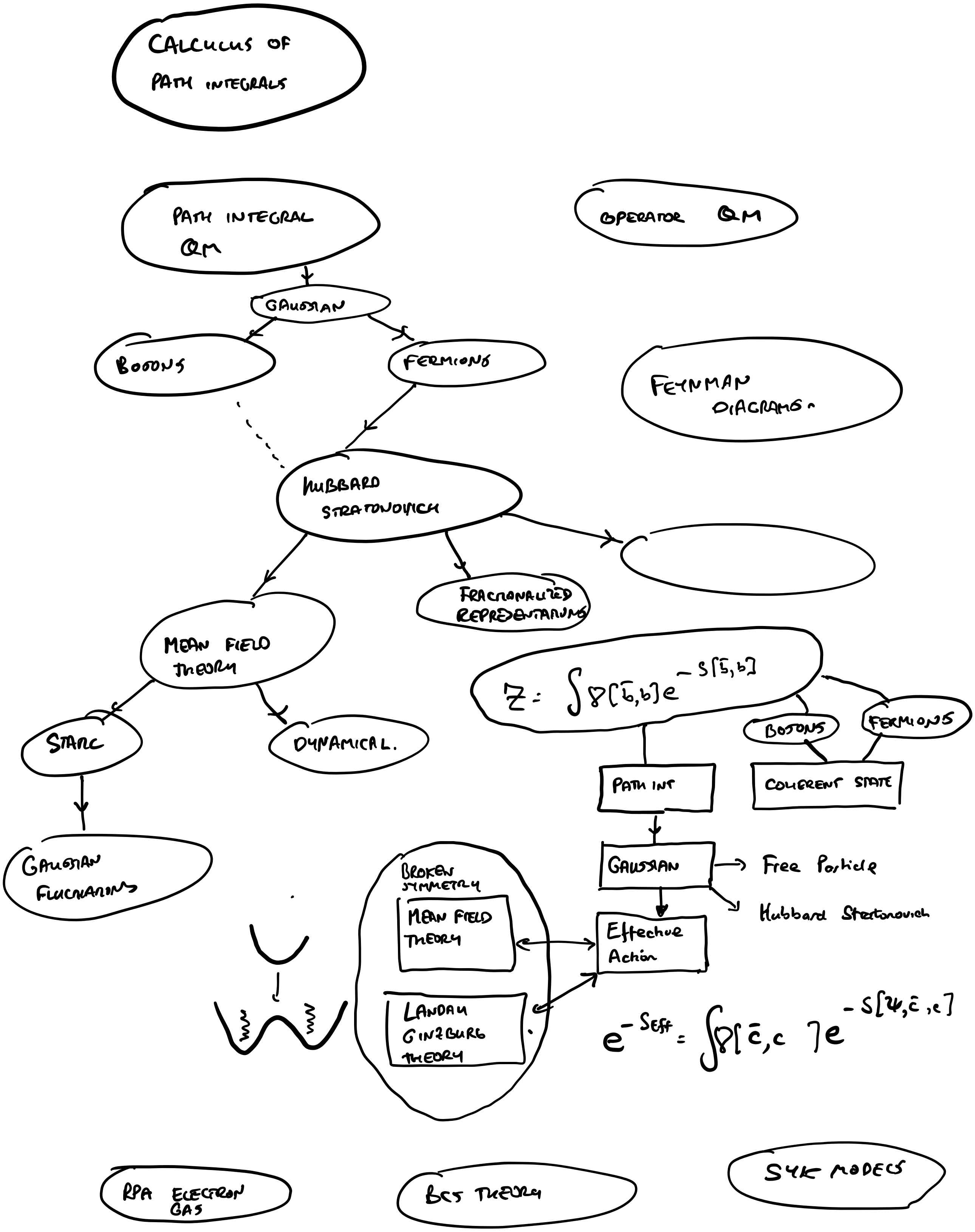
$$\int_0^\beta d\tau \sum \bar c_j \frac{(c_j - c_{j-1})}{\Delta\tau} = \sum_n \bar c(i\omega_n)\underbrace{\frac{\left(1 - e^{i\omega_n \Delta\tau}\right)}{\Delta\tau}}\, c(i\omega_n) \qquad \omega_n = \pi k_B T(2n+1)$$

$$\Delta\tau \to 0 \qquad \bar c(i\omega_n)\ \ -i\omega_n\ \ c(i\omega_n)$$

$$\sum \bar c_n(-i\omega_n + \epsilon)c_n \equiv \int d\tau\ \bar c(\partial_\tau + \epsilon)c$$

$$\frac{1}{\beta}\int d\tau\ \bar c_n(-i\omega_n + \epsilon)c_m\ e^{i(\omega_n - \omega_m)\tau}$$

$$= \sum \underline{\bar c_n(-i\omega_n + \epsilon)}c_n$$


CALCULUS OF PATH INTEGRALS
PATH INTEGRAL QM
OPERATOR QM
GAUSSIAN
BOSONS
FERMIONS
FEYNMAN DIAGRAMS.
HUBBARD STRATONOVICH
FRACTIONALIZED REPRESENTATIONS
MEAN FIELD THEORY
STATIC
DYNAMICAL.
GAUSSIAN FLUCTUATIONS
$Z = \int \mathcal{D}[\bar{b},b]\, e^{-S[\bar{b},b]}$
BOSONS
FERMIONS
PATH INT
COHERENT STATE
GAUSSIAN
Free Particle
Hubbard Stratonovich
Effective Action
BROKEN SYMMETRY
MEAN FIELD THEORY
LANDAU GINZBURG THEORY
$e^{-S_{eff}} = \int \mathcal{D}[\bar{c},c\ ]\, e^{-S[\psi,\bar{c},c]}$
RPA ELECTRON GAS
BCS THEORY
SYK MODELS

$\rightarrow$ Free Particle

$\searrow$ Hubbard Stratonovich

Effective Action

$$e^{-S_{Eff}} = \int \mathcal{D}[\bar{c}, c \ ] \, e^{-S[\psi, \bar{c}, c]}$$