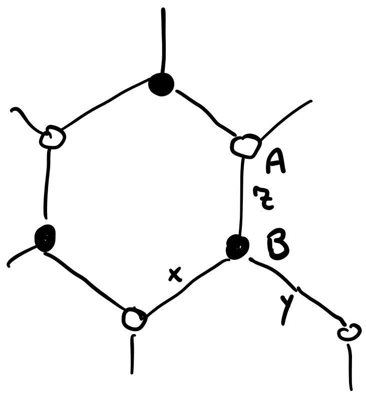
KITAEV MODEL.

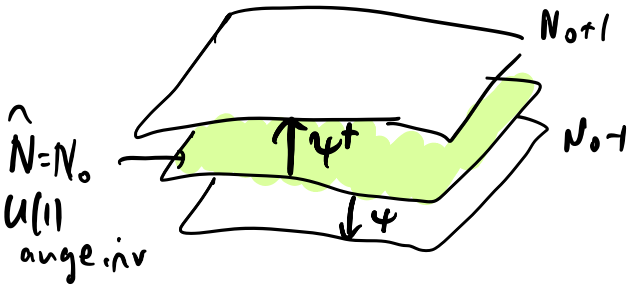
Kitaev, Annals of Physics, 321, 2 (2006).

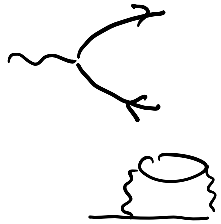
Method 2: Ancilliary Qubits



$$H = -\left(\frac{1}{2}\right) \sum_{\langle i,j \rangle} K_{\alpha_{ij}} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}}$$

The Hilbert space of the spins can be considered to be a slice through the Fock-Space of fractionalized excitations.

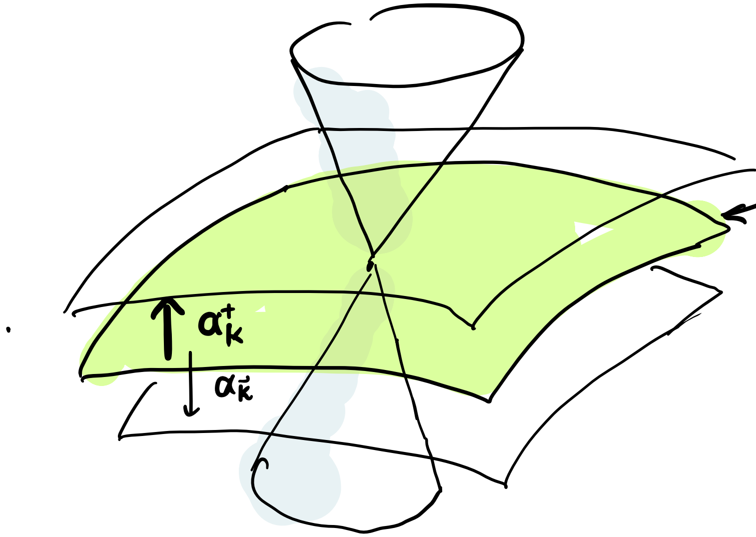




The description of many body physics in terms of field theory relies on expanding a Hilbert space of definite particle number to a Fock space, in which the conservation of charge is protected by gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \varphi$$

$$0 = \int \frac{\delta S}{\delta A_\mu} \partial_\mu \varphi = \int J^\mu \partial_\mu \varphi = -\int \varphi \, \partial_\mu J^\mu$$



Physical Hilbert Space

$$|\psi_{phys}\rangle = \prod_j \tfrac{1}{2}(1+D_j)|\psi\rangle$$

$$D_j = -2i\,\alpha_j \beta_j$$

To each site, we add an ancillary qubit $\quad f^\dagger = \frac{\alpha - i\beta}{\sqrt{2}}\,,\quad f = \frac{\alpha - i\beta}{\sqrt{2}}\,,$

$$f^\dagger f - \tfrac{1}{2} = -i\alpha\beta \qquad D_j = -2i\alpha_j\beta_j = 1 \quad \text{(choice)}$$

$$\beta_j\,\vec{\sigma}_j = \vec{b}_j \qquad \Rightarrow \qquad \vec{\sigma}_j = 2\beta_j \vec{b}_j = -i\vec{b}_j \times \vec{b}_j$$

$$\{b_j^\alpha, b_l^\delta\} = \delta^{\alpha\delta}\delta_{jl} \qquad \beta_j = -2i\, b_j^1\, b_j^2\, b_j^3$$

(check $-2i b^1 b^2 b^3 = -2i\beta_j^3 \overbrace{\sigma_j^x \sigma_j^y \sigma_j^z}^{i}$
$= \beta_j$)

Unfortunately β_j is not independent of the b_j^α. However in the physical slice where $D_j = 1$, $-i\alpha_j = \beta_j$, i.e

$$\boxed{\vec{\sigma}_j = -2i\,\alpha_j\,\vec{b}_j}$$

$$H = \frac{1}{2}\sum_{\langle i,j\rangle} k^{\gamma_{ij}}\,\sigma_i^{\gamma_{ij}}\sigma_j^{\gamma_{ij}} = +2\sum_{\langle i,j\rangle} k^{\gamma}\,(\alpha_i b_i^{\gamma_{ij}})(\alpha_j b_j^{\gamma_{ij}})$$

$\gamma_{ij} = \{x, y, z\}$

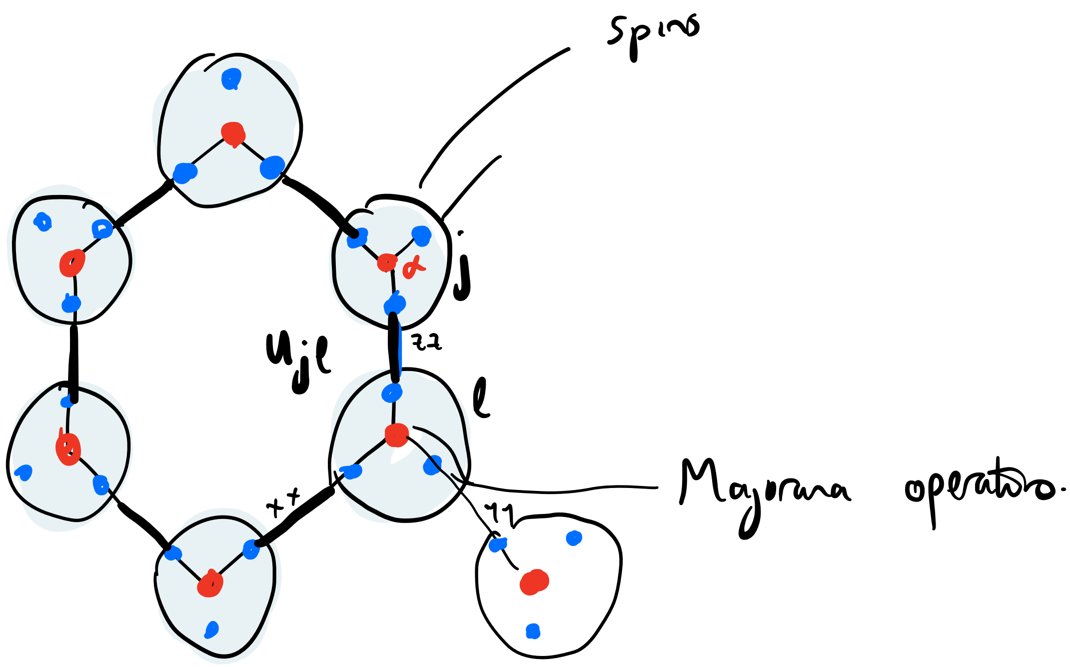
$$= -2\sum k^{\gamma_{ij}}\,(\alpha_i\alpha_j)\,b_i^{\gamma_{ij}} b_j^{\gamma_{ij}}$$

$$\boxed{\begin{gathered} H = \sum_{\langle i,j\rangle} K^{\gamma_{ij}}\,(i a_i a_j u_{ij}) \\ u_{ij} = +2i\,b_i^{\gamma_{ij}} b_j^{\gamma_{ij}} = \pm 1 \quad \text{GAUGE FIELD} \end{gathered}}$$

$$D_j = -2i\alpha_j\beta_j = -4\alpha_j b_j^1 b_j^2 b_j^3 = 1$$

$$\left.\begin{aligned} a_i &\longrightarrow z_i a_i\,, \\ u_{ij} &\longrightarrow z_i u_{ij} z_j\,, \end{aligned}\right\} \quad z_i = \pm 1$$

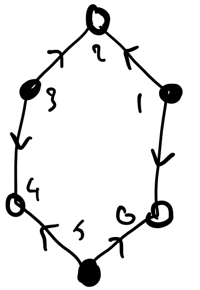
Z_2 GAUGE INVARIANCE.



Spins
j
α
u_{jl}
zz
l
xx
yy
Majorana operators.

The product

$$\hat{W}_p = \prod_{(e+1,e)\in p} (\hat{u}_{e+1,e})$$



$$= u_{12}\, u_{32}\, u_{34}\, u_{54}\, u_{56}\, u_{16}$$

$$-2i b_6^x b_1^x$$

$$= -u_{12} u_{23} u_{34} u_{45} u_{56} u_{61}$$

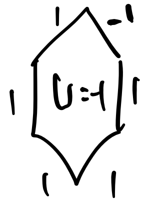
$$= 2^6\, b_1^y b_2^y\, b_2^x b_3^x\, b_3^z b_4^z\, b_4^y b_5^y\, b_5^x b_6^x\, b_6^z b_1^z$$

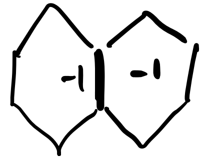
$$= -2^6 (b_1^z b_1^y)(b_2^y b_2^x)(b_3^x b_3^z)(b_4^y b_4^y)(b_5^x b_5^x)(b_6^z b_1^z)$$

$$-2^6 \left(\frac{-i\sigma_1^x}{2}\right)\left(\frac{-i\sigma_2^z}{2}\right)\left(\frac{-i\sigma_3^x}{2}\right)\left(\frac{-i\sigma_4^x}{2}\right)\left(\frac{-i\sigma_5^z}{2}\right)\left(\frac{-i\sigma_6^y}{2}\right)$$

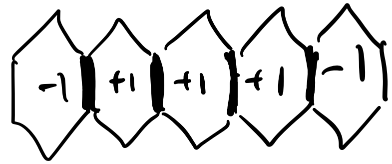
$$= \prod_{j\in p} \sigma_j^{a_j} \qquad [W_p, H] = 0$$

Ground state energy is purely a functional of the W_p, infact $W_p = +1$ defines the GS.



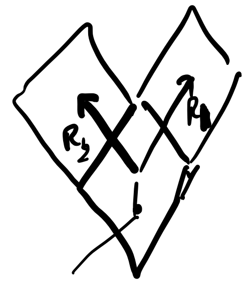


Two Visons



Two visons, separated.

In the GS all $W_p = +1$.

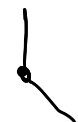


$$R_2 = \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$

$$R_1 = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$

$$H = \frac{1}{2} \sum K^{\alpha_{ij}} \left(\gamma(R_i - R_j)\, \alpha_A(i)\, \alpha_B(j) + h.c. \right)$$

$$\gamma(R) = i\left(K^z \delta_{\vec{R},0} + K^x \delta_{R,R_1} + K^y \delta_{R,-R_2} \right)$$



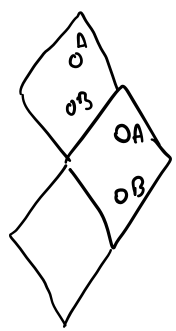
$$\langle k|R\rangle\, \gamma(R)\, \langle 0|k\rangle = \sum_{R} \gamma(R)\, e^{+i\vec{k}\cdot\vec{R}_j}$$

$$\alpha_k = \frac{1}{\sqrt{N}} \sum_j \begin{pmatrix} \alpha_A(j) \\ \alpha_B(j) \end{pmatrix} e^{-i\vec{k}\cdot\vec{R}_j} = \begin{pmatrix} \alpha_{kA} \\ \alpha_{kB} \end{pmatrix}$$

$$\begin{pmatrix} \alpha_A(j) \\ \alpha_B(j) \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{k \in \hexagon} \alpha_k e^{i\vec{k}\cdot\vec{R}_j}$$

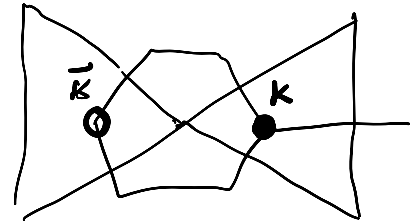
$$= \frac{1}{\sqrt{N}} \sum_{k \in \triangleleft} \left(\alpha_k e^{i\vec{k}\cdot\vec{R}_j} + \alpha_k^{\dagger} e^{-i\vec{k}\cdot\vec{R}_j} \right)$$

$$H = \frac{k}{2} \sum_{k \in \hexagon} \alpha_{\vec{k}}^{\dagger} \begin{pmatrix} 0 & \gamma_k \\ \gamma_k^{\dagger} & 0 \end{pmatrix} \alpha_{\vec{k}}$$



$$\langle k|R\rangle \, \gamma(R) \, \langle 0|k\rangle = \sum_R \gamma(R) e^{-i\vec{k}\cdot\vec{R}_j}$$

$$H = \frac{1}{2} \sum_{k \in \hexagon} \alpha_{\vec{k}}^{\dagger} \begin{pmatrix} 0 & \gamma_k \\ \gamma_k^* & 0 \end{pmatrix} \alpha_{\vec{k}}$$



$$= \sum_{k \in \triangleleft} \alpha_{\vec{k}}^{\dagger} \begin{pmatrix} 0 & \gamma_k \\ \gamma_k^* & 0 \end{pmatrix} \alpha_{\vec{k}}$$

$$\gamma_{\vec{k}} = i\left[k_z + k_y e^{ik\cdot R_2} + k_x e^{ik\cdot R_1}\right]$$

$$= \sum_{k \in \triangleleft} \alpha_{\vec{k}}^{\dagger} \left(\vec{\gamma}_k \cdot \vec{\tau}\right) \alpha_{\vec{k}}$$

$$= \left(\mathrm{Re}\,\gamma_k, -\mathrm{Im}\,\gamma_k, 0\right)$$

$$(\gamma_k \cdot \tau) \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix} = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix} \begin{pmatrix} |\gamma_k| & \\ & -|\gamma_k| \end{pmatrix}$$

$$\gamma v = |\gamma| u$$
$$\gamma^* u = |\gamma| v$$

$$v_k = \gamma^* / \sqrt{2}\,|\gamma_k|$$

$$u_k = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} & \gamma \\ \gamma^* & \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{\gamma^*}{|\gamma|\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{|\gamma|^2}{|\gamma|\sqrt{2}} \\ \frac{\gamma^*}{\sqrt{2}} \end{pmatrix} = |\gamma| \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{\gamma^*}{\sqrt{2}\,|\gamma|} \end{pmatrix}$$

$$\alpha_k = \overbrace{\begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}}^{U} \begin{pmatrix} \eta_{k1} \\ \eta_{k2} \end{pmatrix}$$

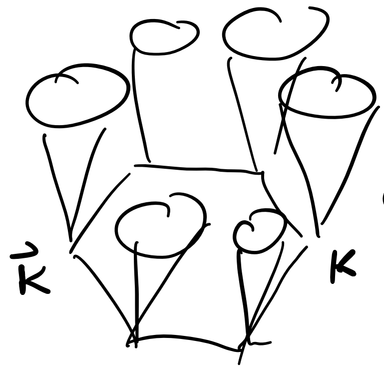
$$H = \sum_{k \in \triangleleft} \epsilon(k) \left[\eta_{k1}^{\dagger} \eta_{k1} - \eta_{k2}^{\dagger} \eta_{k2}\right]$$

$$\alpha = U\eta \qquad \eta_k^{\dagger} = \alpha^{\dagger} U_k$$

$$\boxed{|\Psi_{gs}\rangle = \prod_j \left(\frac{1+D_j}{2}\right) \prod_{k\in\triangleleft} \eta^\dagger_{kz} |\phi\rangle}$$

$$K^x = K^y = K^z = K$$

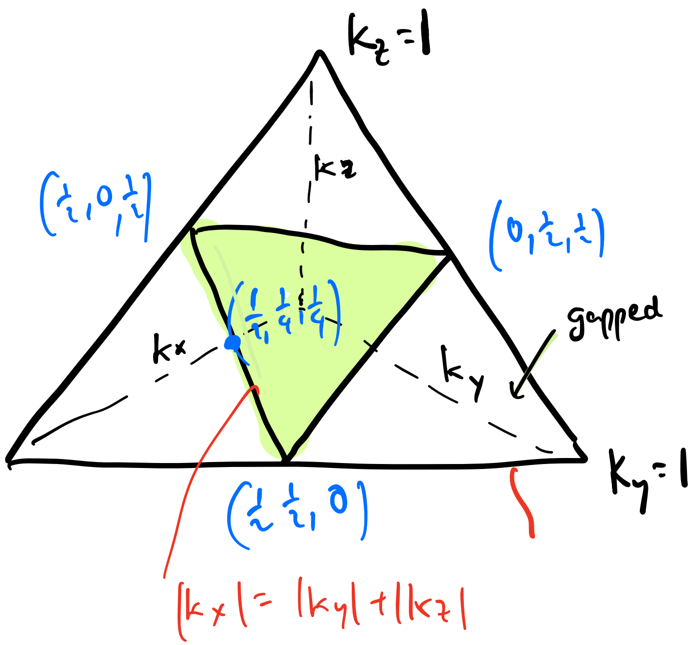
$$e(k) = K\left|1 + e^{i\vec{k}\cdot\vec{R}_1} + e^{i\vec{k}\cdot\vec{R}_2}\right|$$



$e = ck$

$c = \frac{3K}{2}$

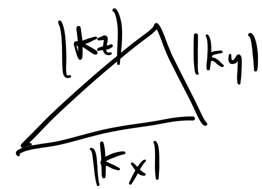
$$\left[k_z + k_x e^{ik\cdot R_1} + k_y e^{ikR_2}\right] = 0$$



I $|k_x| \le |k_y| + |k_z|$

$|k_y| \le |k_x| + |k_z|$

$|k_z| \le |k_x| + |k_y|$



$k_x + k_y + k_z = 1$

$k_x = 1 - 2k < 2k$

$4k > 1$

$k > 1/4$

$k_x = k_y = k_z = K$

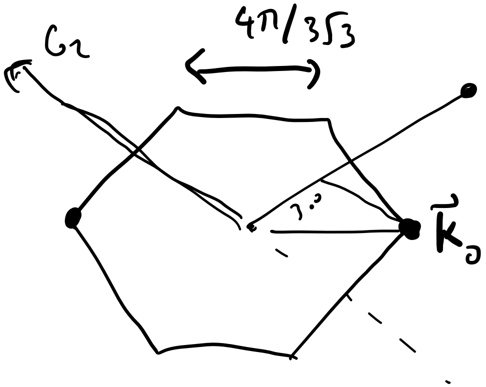
$\gamma(k) = ik\left(1 + e^{i\vec{k}\cdot R_1} + e^{i\vec{k}\cdot\vec{R}_2}\right)$

$k_x = k_y = k_z = K$

$\vec{k} = k_1\vec{G}_1 + k_2\vec{G}_2$

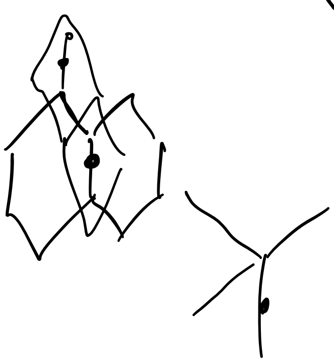
$\vec{k}\cdot R_1 = 2\pi k_1 = 2\pi/3$

$\vec{k}\cdot\vec{R}_2 = 2\pi k_2 = -2\pi/3$



$\vec{k} = \frac{2}{3}G_2 + \frac{1}{3}G_1$

$-\vec{k} = -\frac{2}{3}G_2 - \frac{1}{3}G_1 \equiv \frac{1}{3}G_2 + \frac{2}{3}G_1$



$n_1 = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$n_2 = \left(\frac{-\sqrt{3}}{2}, \frac{3}{2}\right)$

$\vec{G}_1 = \frac{2\pi}{3}\left(\sqrt{3}, 1\right)$

$\vec{G}_2 = \frac{2\pi}{3}\left(-\sqrt{3}, 1\right)$

$\alpha\frac{2\cdot 3}{2} = 2\pi$

$\vec{k}_0 = \frac{1}{3}(G_1 - G_2)$

$= \frac{4\pi}{3\sqrt{3}}(1, 0)$

$G_1\cdot n_1 = \frac{2\pi}{3}\left(\frac{3}{2} + \frac{3}{2}\right) = \frac{2\pi}{3}\cdot 3 = 2\pi$ ✓

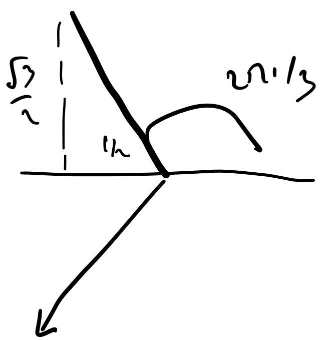
$\frac{2\pi}{2\cdot 3}(3+3) = 2\pi$

Velocity

$$\gamma(\vec{k}_0 + \delta k) = \nabla_k \gamma \cdot \delta\vec{k}$$

$$\nabla_k \gamma = i k \left(i\vec{R}_1 e^{i\vec{k}_0\cdot R_1} + iR_2 e^{i\vec{k}_0\cdot R_2} \right)$$

$$= -K \left(e^{i\vec{k}_0\cdot\vec{R}_1}\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) + e^{i\vec{k}_0\cdot R_2}\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)\right)$$



$$= -K \left(e^{2\pi i/3}\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) + e^{-2\pi i/3}\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)\right)$$

$$= -k \left(\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)\right)$$

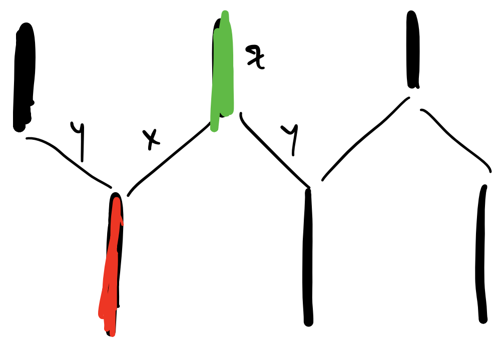
$$= \frac{3}{2} k \left(-i, 1\right)$$

$$\gamma(k) \simeq \frac{3k}{2}\left(-i\delta k_x + \delta k_y\right)$$

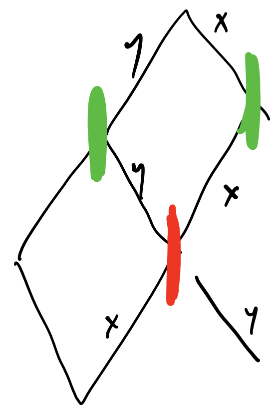
$$\epsilon(k) = \frac{3k}{2}\sqrt{\delta k_x^2 + \delta k_y^2} \Rightarrow \boxed{c = \frac{3k}{2} \qquad \epsilon = c\,\delta k}$$

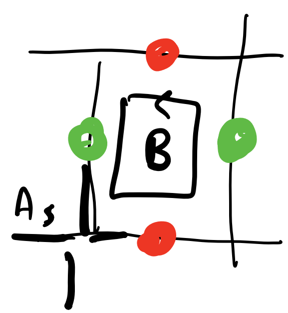
Gapped Phase : Anyons

$$|K_x| + |K_y| \leq |K_z|$$



$|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$





$$H_{eff} = -J_{eff}\left(\sum_{\text{vertices}} A_s + \sum_{\text{plaquettes}} B_p\right)$$

$$A_s = \prod_{\text{stars}} \sigma_j^x \qquad B_s = \prod_{\text{bonds}} \sigma_j^z$$

$\boxed{\text{Toric code}}$

$$J_{eff} = -J_x^2 J_y^2 / 16 |J_z|^3 .$$