Spin liquids and Gauge Theories

The spin liquid concept is intimately connected with the idea of "fractionalization". The Hilbert space of a single spin is simply $\{\uparrow, \downarrow\}$; it is two dimensional. Anderson envisaged that a spin liquid could be thought of as an incompressible fluid of $S=1/2$ fermions, described by a Gutzwiller wavefunction e.g we could consider

$$|\psi_{RVB}\rangle = P_G \prod_{k<k_F} f^\dagger_{k\sigma} |\phi\rangle$$

where $P_G = \prod_j (n_{j\uparrow} - n_{j\downarrow})^2$ projects out the states where $n_f = 1$, and the Fermi surface is half filled. We see that the Hilbert space of the excitations is "bigger" than the Hilbert space of spins, but that the projector projects out doubly occupied or empty sites, returning us to the "physical" Hilbert space.

In this description, the spin operator acting on $|\Psi_{RVB}\rangle$ has "fractionalized" into fermionic spinons

$$\vec{S} = f_\alpha^\dagger \frac{\vec{\sigma}_{\alpha\beta}}{2} f_\beta .$$

A spin flip creates a particle hole pair. Now to avoid the unphysical doubly occupied or empty states, we must impose the constraint

$$n_f(j) = 1$$

at each site. Indeed we see that $[\vec{S}_j, n_f(j)] = 0$, so that

if we write a Heisenberg model for an antiferromagnet

$$\mathcal{H} = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = \frac{J}{4} \sum \left(f^{\dagger}_{i\alpha} \vec{\sigma}_{\alpha\beta} f_{i\beta} \right) \cdot \left(f^{\dagger}_{j\gamma} \vec{\sigma}_{\gamma\delta} f_{j\delta} \right).$$

then we arrive at a Hamiltonian that has a conserved quantity

$n_f(j) = f^{\dagger}_{j\sigma} f_{j\sigma}$ at each site. These operators are generators of

a local $U(1)$ gauge symmetry:

$$f_{i\sigma} \to e^{i\phi_i} f_{i\sigma}$$

& if the spins fractionalize into Dirac fermions, we will obtain a U(1) spin liquid.

One way to follow this physics is to rewrite the Heisenberg interaction in the "Coqblin-Schrieffer" form, i.e. using

$M = \alpha_\mu \sigma^\mu = \frac{1}{2} \mathrm{Tr}\, M \sigma^\mu \sigma^\mu$

$\alpha_\mu = \frac{\mathrm{Tr}\, M\sigma}{2}$

$$\sigma^\mu_{\alpha\beta}\, \sigma^\mu_{\delta\gamma} = 2\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}$$

$$\Rightarrow \quad f^\dagger_{i\alpha} \frac{\vec{\sigma}}{2} f_{i\beta}\; f^\dagger_{j\delta} \frac{\vec{\sigma}_{\delta\gamma}}{2} f_{j\gamma} = -\frac{\hat{n}_i \hat{n}_j}{4} + \frac{1}{2} f^\dagger_{i\alpha} f_{i\beta} f^\dagger_{j\beta} f_{j\alpha}$$

$$= -n_i n_j - \frac{1}{2} :f^\dagger_{i\alpha} f_{j\alpha} f^\dagger_{j\beta} f_{i\beta}:$$

$$= -1 - \frac{1}{2} :(f^\dagger_{i\alpha} f_{j\alpha})(f^\dagger_{j\beta} f_{i\beta}):$$

So we can rewrite the Heisenberg Hamiltonian in the form

$$\mathcal{H} = - \frac{J}{2} \sum_{\langle i,j \rangle} :\left(f^{+}_{i\alpha} f_{j\alpha}\right)\left(f^{+}_{j\beta} f_{i\beta}\right): \quad + \sum_{j} \lambda_j \left(n_{fj} - 1\right)$$

The last term is a constraint — λ_j is to be integrated between $\lambda_j = 0$ & $\lambda_j = 2\pi i k_B T$, to impose the constraint

$$P_G = \prod_j \int_0^{2\pi i k_B T} \frac{d\lambda_j}{2\pi i k_B T} \exp\left[-\beta \lambda_j (\hat{n}_{fj} - 1)\right]$$

So we can write the Heisenberg model as

$$Z = \mathrm{Tr}\left[P_G e^{-\beta \mathcal{H}}\right] = \int \mathcal{D}[\bar{f}, f, \lambda] e^{-S}$$

where

$$S = \int d\tau \left[\sum_j f^\dagger_{j\sigma}(\partial_\tau + \lambda_j) f_{j\sigma} - 2J \sum (f^\dagger_{i\sigma} f_{j\sigma})(f^\dagger_{j\sigma'} f_{i\sigma'}) \right.$$
$$\left. - \lambda_j Q_j \right]$$

If we now carry out a Hubbard Stratonovich transformation of the interaction, we obtain

$$-\frac{J}{2} \sum_{\langle i,j \rangle} (f^\dagger_{i\alpha} f_{j\alpha})(f^\dagger_{j\beta} f_{i\alpha}) \rightarrow \sum_{\langle i,j \rangle} \left(f^\dagger_{i\alpha} \Delta_{ij} f_{j\alpha} + f^\dagger_{j\alpha} \bar{\Delta}_{ij} f_{i\alpha} \right.$$
$$\left. + \frac{2 \bar{\Delta}_{ij} \Delta_{ij}}{J} \right).$$

The resulting action

$$S = \int d\tau \Big\{ \sum_j f^\dagger_{j\sigma} (\partial_\tau + \lambda_j) f_{j\sigma} - \lambda_j Q$$

$$+ \sum_{(i,j)} \Big[\Delta_{ij} f^\dagger_{i\alpha} f_{j\alpha} + f^\dagger_{j\alpha} f_{i\alpha} \bar{\Delta}_{ij} + \frac{2 \bar{\Delta}_{ij} \Delta_{ij}}{J} \Big] \Big\}$$

describes a "U(1) gauge theory", which has the following gauge symmetries

$$f_{j\sigma} \to e^{i\phi_j} f_{j\sigma}$$

$$\lambda_j \to \lambda_j - i \partial_\tau \phi_j \qquad U(1) \text{ gauge symmetry.}$$

$$\Delta_{ij} \to e^{i(\phi_i - \phi_j)} \Delta_{ij}$$

Indeed, if we write $\Delta_{ij} = |\Delta_{ij}| e^{-i A_{ij}}$, we see that

$A_{ij} \sim \int_j^i \vec{A} \cdot d\vec{r}$ is a kind of Peierls substitution, in which

$(\lambda, \vec{A})$ form the components of an emergent gauge field.

Thus we see that

$$\boxed{\text{Spin Hilbert Space} \rightarrow \text{Spinon Hilbert space} + (U(1)) \text{ Gauge field}}$$

In such gauge theories there is always the question of

whether the microscopic particles — in this case fermionic spinors,

are confined or deconfined.

We can seek a mean-field description of and a deconfined phase by looking for saddle point descriptions of the ground state. Suppose we make the Ansatz

$$\Delta_{ij} = \text{const} = -\Delta \qquad \lambda_j = \lambda$$

Then our mean field theory is

$$\mathcal{H}_{MFT} = -\sum_{\langle ij \rangle} \Delta \left(f^\dagger_{i\alpha} f_{j\alpha} + h.c. \right) + \lambda \sum \left[n_f(j) - Q \right] + \frac{N|\Delta|^2}{J}$$

$(Q = 1)$

i.e in momentum space $\quad f_{j\alpha} = \frac{1}{\sqrt{N_s}} \sum f_{\vec{k}\alpha} e^{i\vec{k}\cdot\vec{R}_j}$

$|j\rangle = |k\rangle\langle k|j\rangle$

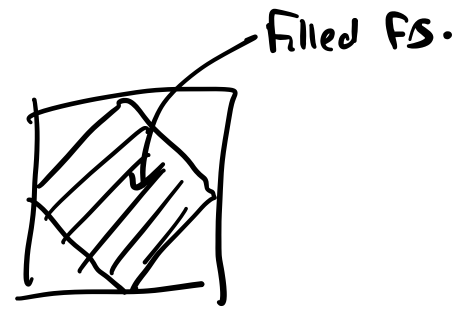
$$H_{MFT} = \sum_{\sigma=1}^{N} \epsilon_k f_{k\sigma}^{+} f_{k\sigma} + N N_s \frac{|D|^2}{J} - \lambda N_s Q .$$

$$\epsilon_k = -2\Delta(\cos k_x + \cos k_y) + \lambda$$

In fact half filling, $Q=1$, is obtained at $\langle n_f\rangle = 1$ if $\lambda = 0$

& we have a ground state that is a filled Fermi sea

$$|\Psi_g\rangle = P_G \prod_{k<k_F} c_{k\uparrow}^{+} c_{k\downarrow}^{+} |\phi\rangle$$



We can try to estimate the ground state energy in the mean-field theory as follows

The ground-state energy is then (for SU(N))

$$\langle \hat{u} \rangle = N \int \frac{d^2k}{(2\pi)^2} \overbrace{\left(\overbrace{-2\Delta\,(c_x + c_y)}^{\epsilon_k} \right)}^{f(\epsilon_k)} + \frac{z\Delta^2 N}{J}.$$

$z = 2 = \#$ of bonds per site

where we've set $Q/N = 1/2$ & $\lambda = 0$ (half filling).

Stationarity w.r.t. variations in Δ gives

$$\frac{\delta \langle u \rangle}{\delta \Delta} = \frac{4\Delta}{J} - \overbrace{\int \frac{d^2k}{(2\pi)^2} 2(c_x + c_y) f_k}^{8/\pi^2}$$

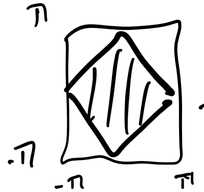
$$\Rightarrow \quad \frac{2\Delta}{J} = \int \frac{d^2k}{(2\pi)^2} (c_x + c_y) f_k$$

$$\int \frac{d^2k}{(2\pi)^2} (c_x + c_y) = \frac{4}{\pi^2}$$

Changing variables to $k_x = u - v \quad k_y = u + v$

$$c_x + c_y = 2\cos u \cos v$$

$$dk_x\,dk_y = du\,dv \left\| \frac{\delta[k_x, k_y]}{\delta[u, v]} \right\| = du\,dv \left\| \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \right\| = 2\,du\,dv$$



$$\frac{2\Delta}{J} = \int \frac{d^2k}{(2\pi)^2} (c_x + c_y) f_k = \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} du\,dv \, \cos u \cos v = \frac{4}{\pi^2} \Rightarrow \Delta = \frac{2J}{\pi^2}$$

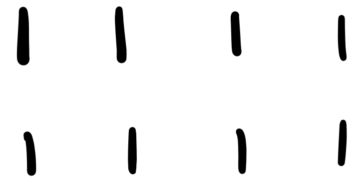
$$\frac{\langle H \rangle}{N} = \frac{E_g}{N} = -2\Delta \underbrace{\int \frac{d^2k}{(2\pi)^2} (c_x + c_y) f_k}_{4/\pi^2} + \frac{2\Delta^2}{J}$$

$$= -\frac{8}{\pi^2}\Delta + \frac{2\Delta^2}{J}$$

$$\boxed{\frac{E_g}{N} = \frac{-16J}{\pi^2} + \frac{8J}{\pi^2} = -\frac{8J}{\pi^2}}$$

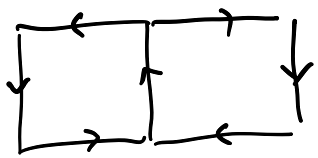
Actually, can try some other phases

Dimer phase



Flux phase

$$\prod \Delta_{ij} = -|\Delta|^4$$

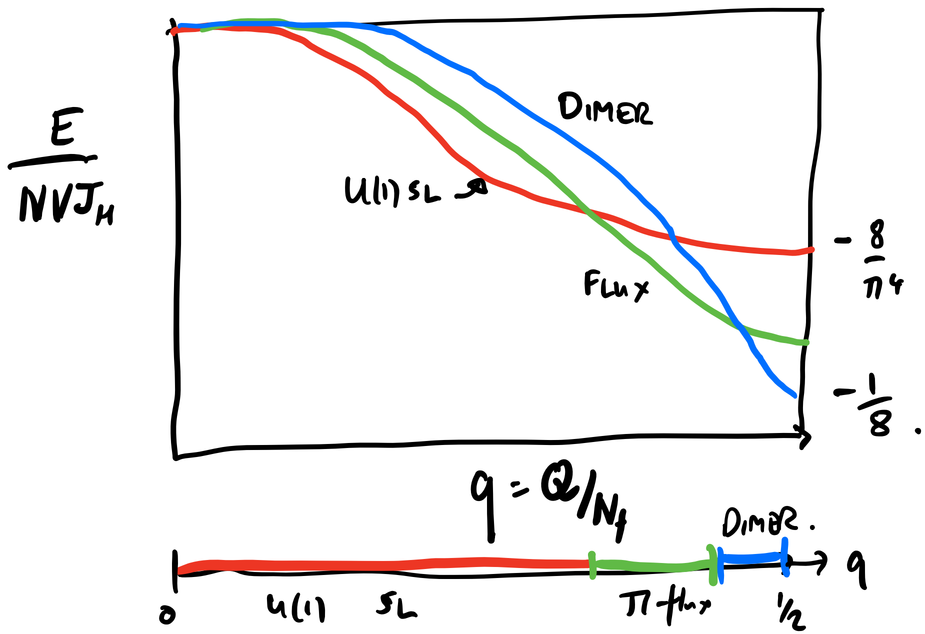


$$\Delta = |\Delta| e^{i\pi/4}.$$

Large N

$$\boxed{H = -\frac{J_H}{N} \sum (f_i^\dagger f_j)(f_j^\dagger f_i) + \lambda(n_i - Q).}$$

$$q = Q/N.$$



In fact none of these mean field phases is relevant to the 2D Heisenberg model, which magnetically orders. However, if we look at the large N model, described by

Large N − SU(N) Heisenberg model

$$\boxed{\mathcal{H} = -\frac{J_H}{N} \sum (f_i^\dagger f_j)(f_j^\dagger f_i) + \lambda(n_i - Q).}$$

$n_{fi} = Q$ at each site

SU(N)

then it is possible to stabilize various flux/spin liquid phases

To truly stabilize these phases probably requires a plaquet term

$$\mathcal{H} \to \mathcal{H} - k \sum_{\square} \prod_{\square} \Delta_{ij}$$

To avoid the difficulties of the U(1) spin liquid, we will examine a closely related class of spin liquids called $\mathbb{Z}_2$ spin liquids, in which the underlying gauge theory the gauge transformations are discrete. About 20 years ago, Alexei Kitaev discovered a family of exactly solvable $\mathbb{Z}_2$

Spin liquids which we will now examine.