MANY BODY PHYSICS: 621. Spring 2024

Exercise 3. Toric Code and Twisted Bilayer Graphene (Due May 6th. Pdf solutions by email subject "621 Homework 3" please

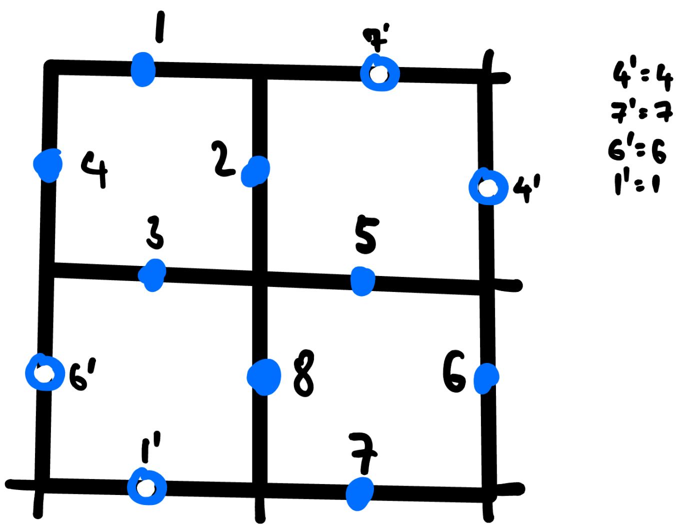

FIG. 1: Toric code on a four-plaquet, four star torus.

1. The Toric code is given by

$$H = -J_e \sum_S A_S - J_M \sum_P B_P \tag{1}$$

where $A_S = \prod_{j \in +} X_j^x$ is the product of x-spins $X_j \equiv \sigma_j^x$ around a "star" while $B_P = \prod_{j \in \square} Z_l$ is the product of the Pauli-z spin $Z_j \equiv \sigma_j^z$ around a plaquet. Consider the Toric code on a torus with eight sites, four stars and four plaquets as shown in Fig. 1, Denote the state with all spins down by $\Psi_0 = |0_1 0_2 0_3 0_4, 0_5 0_6 0_7 0_8\rangle$, where the labelling of the spins is shown in Fig. 1.

   (a) Write down the ground-state wavefunction obtained by summing over all states obtained by flipping even numbers of spins around each star, i.e

$$\Psi_g = (1 + A_s(1))(1 + A_s(2))(1 + A_s(3))(1 + A_s(4))(1 + |\Psi_0\rangle$$

   What is the ground-state energy of this state?

(b) Construct the three other topologically degenerate ground-states, by acting on the ground-state you have already found with a "magnetic" (product of X operators around torus).

2. (a) The four node, 3 $\mathbf{q}$ Bistritzer-MacDonald Hamiltonian is

$$H(\mathbf{k}) = \begin{pmatrix} h_{\mathbf{k}}(-\theta/2) & T_1 & T_2 & T_3 \\ T_1^\dagger & h_{\mathbf{k}+\mathbf{q}_1}(\theta/2) & & \\ T_2^\dagger & & h_{\mathbf{k}+\mathbf{q}_2}(\theta/2) & \\ T_3^\dagger & & & h_{\mathbf{k}+\mathbf{q}_3}(\theta/2) \end{pmatrix} \tag{2}$$

where, putting

$$h_{\mathbf{k}}(\theta) = v \begin{pmatrix} & \bar{k}e^{i\theta} \\ ke^{-i\theta} & \end{pmatrix}, \qquad T_l = w \begin{pmatrix} 1 & e^{-i(l-1)\phi} \\ e^{i(l-1)\phi} & 1 \end{pmatrix}, (l = 1, 2, 3; \phi = 4\pi/3). \tag{3}$$

where $k = k_x + ik_y$ $\bar{k} = k^*$ and $q_l = -ik_\theta e^{i(l-1)\phi}$. Set this Hamiltonian up in mathematica, (see example in Fig.2) or your favorite coding language, and plot the velocity of the Dirac cones versus the quantity $\alpha = (w/vk_\theta)$. (You may set $\theta = 0$ in the $h_{\mathbf{k}}$ terms.) Confirm that the results for small α correspond to the perturbation theory. (n.b. $w \sim 110$meV and $v_D k_D \sim 11.2$eV in graphene). (Hint: Evaluate the eigenvalues for some small momentum, eg $k = 0.005$, sort the eigenvalues and select the 5th one and divide by k.)

(b) Set up the Hamiltonian for the six site (twelve band) Bistritzer Macdonald model with C_3 symmetry, i.e a six node set-up arranged around the corners of a hexagon. (Hint: the Hamiltonian for the six sites will be $h_{\mathbf{k}-\mathbf{K}_l}((-1)^l\theta/2)$, where in complex notation, $\mathbf{K}_l = q_\theta e^{(2l-1)i\alpha}$ is the location of the lth Dirac cone, where $\alpha = \pi/6$.)

(c) Plot the dispersion across the Brillouin Zone in this simple model.

First set up the sub - matrices

```
In[∘]:= h[k_, θ_] := {{0, Conjugate[k ] Exp[I θ]}, {k Exp[-I θ], 0}};
MatrixForm[h[kx + I ky, ϕ]]
ϕ = 2 π / 3; q = {-I , -I Exp[I ϕ], -I Exp[2 I ϕ] };
T[α_] = α Table[( 1          Exp[-I l ϕ]
                  Exp[I l ϕ]  1          ), {l, 0, 2}];
Table[MatrixForm[T[a][[i]]], {i, 1, 3}]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & e^{\frac{2 i \pi}{3}} (\text{Conjugate}[kx] - i\, \text{Conjugate}[ky]) \\ e^{-\frac{2 i \pi}{3}} (kx + i\, ky) & 0 \end{pmatrix}$$

Out[∘]=

$$\left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} a & a\, e^{-\frac{2 i \pi}{3}} \\ a\, e^{\frac{2 i \pi}{3}} & a \end{pmatrix}, \begin{pmatrix} a & a\, e^{\frac{2 i \pi}{3}} \\ a\, e^{-\frac{2 i \pi}{3}} & a \end{pmatrix} \right\}$$

Now build the Full Hamiltonian, making use of the command “ArrayFlatten”

```
In[∘]:= HH[k_, Θ_, α_] =
  ArrayFlatten[( h1  T1  T2  T3
                 T1  h2  0   0
                 T2  0   h3  0
                 T3  0   0   h4 ) /. {h1 -> h[k, -Θ / 2], h2 -> h[k + q[[1]], Θ / 2],
      h3 -> h[k + q[[2]], Θ / 2], h4 -> h[k + q[[3]], Θ / 2], h10 ->
       h[k + q[[3]] - q[[2]], -Θ / 2], T1 -> T[α][[1]], T2 -> T[α][[2]], T3 -> T[α][[3]]}];

In[∘]:= w0 = 0.110; hbar = 1.054 × 10^-34;
ee = 1.602 × 10^-19;
aa = 1.42 × 10^-10; kD = 4 π / (3 √3 aa); vF = 10^6; tt = hbar vF kD / ee
α0 = w0 / tt
(3 w0 / tt) 360 / (2 π);

Out[∘]= 11.2052

Out[∘]= 0.00981691
```

Now calculate the velocity as a function of inverse twist angle.

FIG. 2: Example set up of mathematica code.