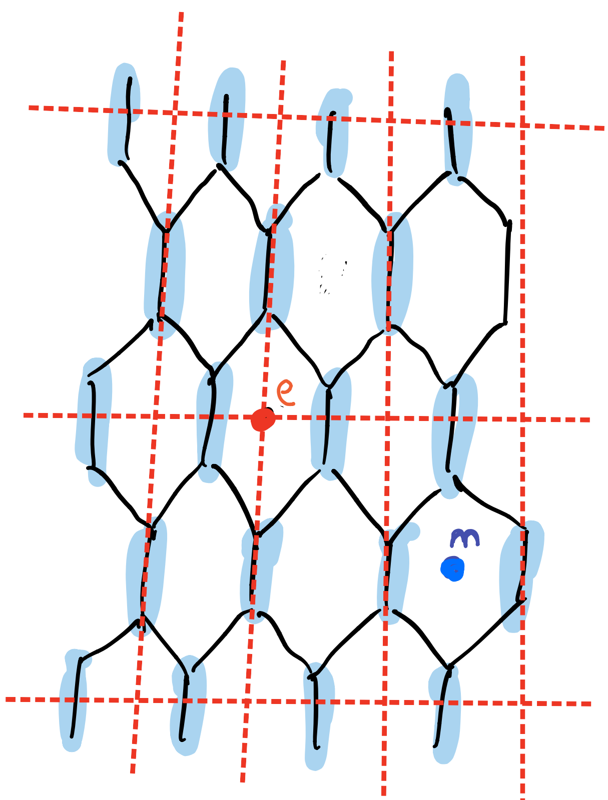
$$|\psi^{M}_{p_1 p_2}\rangle = W^{M}_{\ell^*}|\psi_0\rangle \qquad \Delta E = 2J_m$$

In the Honeycomb model, these excitations are degenerate because $J_m = J_e = \dfrac{J_x^2 J_y^2}{16 J_z^3}$.



$$e: \prod_{+} \sigma_j^x = -1$$

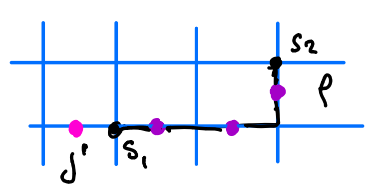
$$m: \prod_{\square} \sigma_j^z = -1$$

"Abelian Anyons"

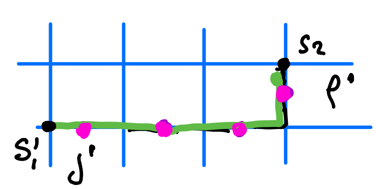
ABELIAN ANYONS : BRAIDING, SUPERSELECTION & FUSION.

The anyons of the Toric code are examples of "Abelian" anyons. When we braid them, the braiding processes commute. We can "move" the e & m anyons by simply extending the string creation operators, e.g

$$W^{E}_{e'} = \sigma^{z}_{j'} \prod_{j\in e} \sigma^{z} = \prod_{j\in e'} \sigma^{z}_{j}$$

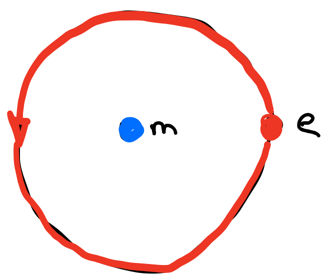


W^{E}_{e}

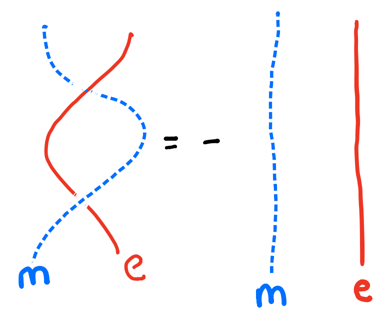


$W^{E}_{e'}$

If we take an e around an m, we pick up a string of $\prod_{\square \atop \text{loop}} \sigma^z = -1$, so that moving an e around an m yields -1.

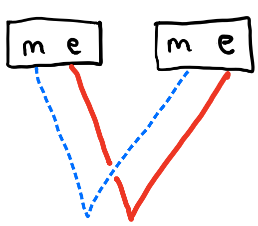


$= -1$,

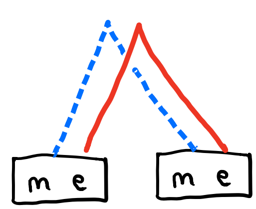


Similarly taking e's around e's produce $+1$, likewise with m's, so they are bosons to themselves. However the combination of an m & an e, which is called an ε behaves as a fermion!

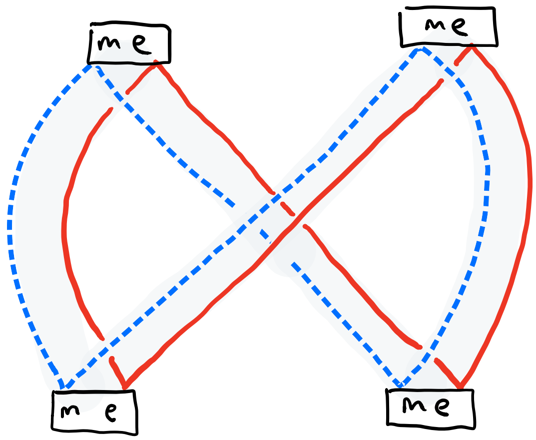
creation =



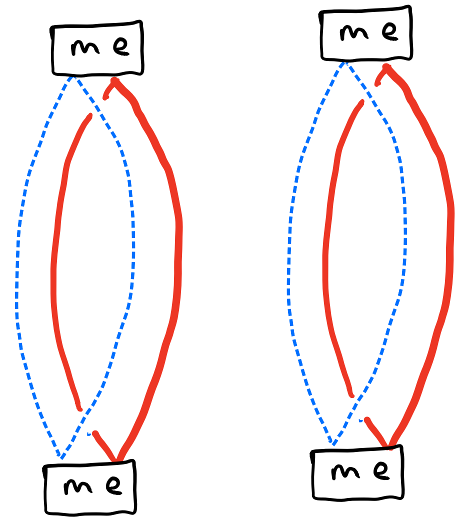
annihilation =

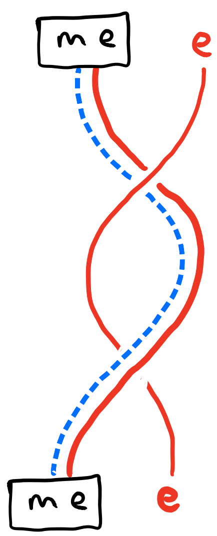


$\varepsilon \equiv$ Fermion

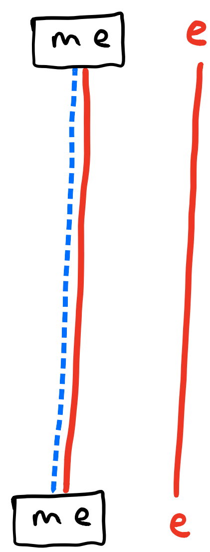


$= -$





$= -$



SUPER-SELECTION

Class of states that can be transformed into one-another by local operators. e.g the current operator can fuse a positron + electron into a photon.

$$e^+ \times e^- = \gamma$$

TORIC CODE : 1 (vacuum), e, m, ε.

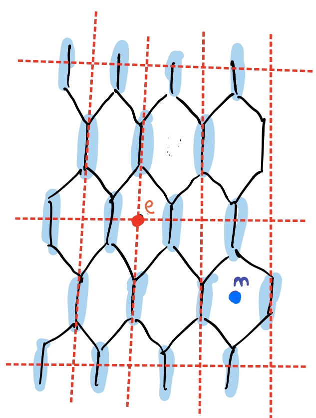
Fusion Rules: How particles (or "superselection sectors") fuse together.

For the toric code

$$e \times e = m \times m = \varepsilon \times \varepsilon = 1 .$$

$$e \times m = \varepsilon, \quad e \times \varepsilon = m, \quad m \times \varepsilon = e .$$

Identification of Particles in the Honeycomb Lattice.

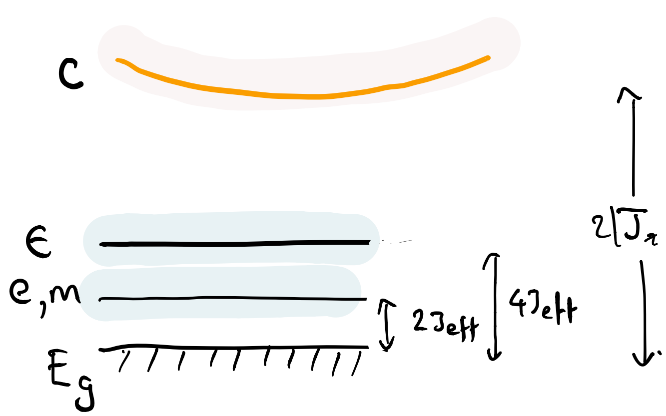


Vortices $\equiv$ e & m

$\boxed{\text{Majorana fermions } c} \equiv \varepsilon$ superselection sector.

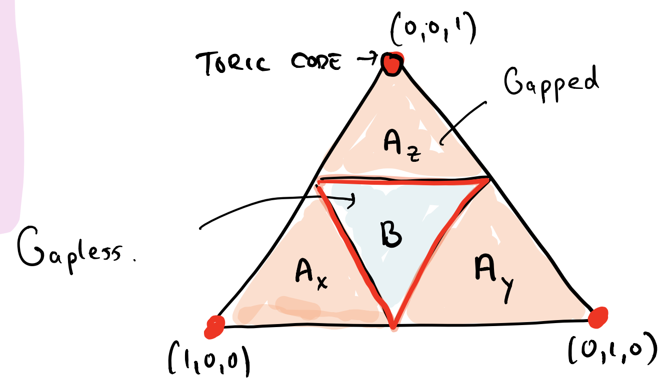
In limit $J_z \gg J_x, J_y$,

$$E(c) \sim 2|J_z|$$

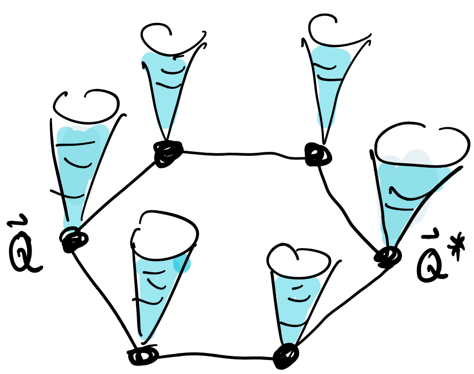


$$E(e\text{-}m) = 4J_{eff} \ll |J_z|$$

Phase B in a Magnetic Field.



In phase B, the gapless fermions prevent the vortices from being moved adiabatically, so they no longer have well defined statistics.



$|Q-Q^*| \sim L^{-1}$

Leads to a kind of RKKY interaction between vortices that is of order $\epsilon(q \sim L^{-1}) \sim L^{-1}$ that oscillates at wavevector

$Q^* - Q \equiv 2Q^*$. This produces a non universal phase

$\Delta\phi \sim L^{-1} t \sim \frac{1}{\text{velocity}}$. As $v \to 0$, $\Delta\phi \to$ very large

$\therefore$ no well defined statistics

But! Broken time reversal symmetry can induce a gap in the Fermion spectrum!

$$V = -\sum_j \vec{h}\cdot\vec{\sigma}_j \qquad \left(\frac{1}{E_0-\mathcal{H}}\right)' = (1-P)\frac{1}{E_0-\mathcal{H}}(1-P)$$

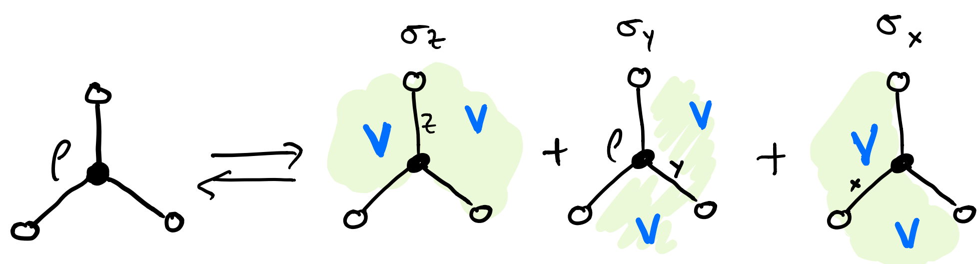
$$H_{eff} = P\Big[\underbrace{V}_{=0} + \underbrace{V\left(\frac{1}{E_0-\mathcal{H}}\right)'V}_{\text{Preserves T-rev}} + V\left(\frac{1}{E_0-\mathcal{H}}\right)'V\left(\frac{1}{E_0-\mathcal{H}}\right)'V + \dots\Big]P$$

(Poor Man's R.G)

V creates two adjacent vortices.

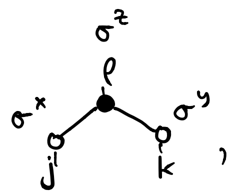
$$\left(\frac{1}{E_0-\mathcal{H}}\right)' \sim -\frac{1}{J} \qquad V = \sum -i\left(h^x b_j^x + h^y b_j^y + h^z b_j^z\right)c_j$$

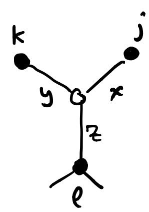
FLIPS



$$\boxed{H_{eff} \sim -\frac{h_x h_y h_z}{J^2} \sum_{j,k,\ell} \sigma_j^x \sigma_k^y \sigma_\ell^z}$$

& symmetry equivalents.





$\sigma_j^x \sigma_\ell^z \sigma_k^y$

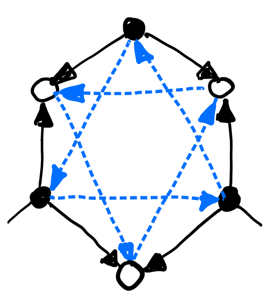
six fermion term

$$\sim -i\, b_j^x c_j\, b_\ell^z c_\ell\, b_k^y c_k$$

$$\sim +i\, c_j \underbrace{b_j^x b_\ell^x}_{u_{j\ell}} \underbrace{(b_\ell^x b_\ell^z b_\ell^y c_\ell)}_{D_\ell} \underbrace{b_\ell^y b_k^y}_{u_{\ell k}} c_k$$

NEXT NEAREST NEIGHBOR HOPPING!

$$= i(D_\ell\, \hat{u}_{j\ell}\, \hat{u}_{\ell k})\, c_j c_k \sim -i\, c_j c_k .$$

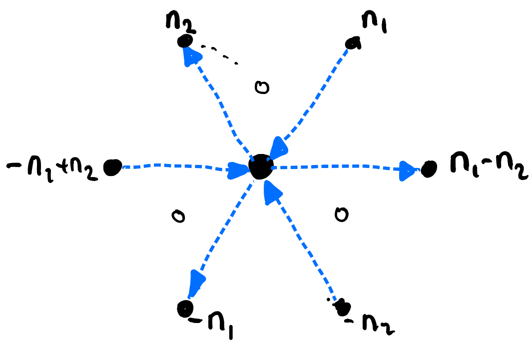


$$H_{eff} = \frac{i}{4} \sum A_{jk} c_j c_k$$

$$A = 2J(\leftarrow) + 2K(\dashleftarrow)$$

$$K \sim \frac{h_x h_y h_z}{J^2}$$

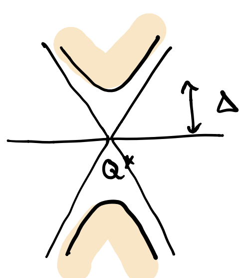
$$H_{eff} = \sum_{k \in \frac{1}{2}BZ} \psi_k^\dagger \, iA(\vec{q}) \, \psi_k \qquad iA(\vec{q}) = \begin{pmatrix} \Delta(q) & if(q) \\ -if^*(q) & -\Delta(q) \end{pmatrix}$$



$$\Delta(q) = i2K\left(e^{i\vec{q}\cdot\vec{n}_2} + e^{-i\vec{q}\cdot\vec{n}_1} + e^{i\vec{q}\cdot(\vec{n}_1-\vec{n}_2)} - h.c\right)$$

$$= 4K\left(\sin(\vec{q}\cdot\hat{n}_1) - \sin(\vec{q}\cdot n_2) + \sin[\vec{q}\cdot(\vec{n}_1-\vec{n}_2)]\right)$$

$$\Delta = \Delta(Q^*) \sim K \sim \frac{h_x h_y h_z}{J^2} \qquad \underline{\underline{\text{GAP}}}$$



$$\epsilon(\vec{q}) \sim \pm\sqrt{3J^2\,\delta q^2 + \Delta^2}.$$

The gapped spectrum has a non zero Chern number.

$$A_{\vec{k}} = i\,\psi_k^* \nabla_k \psi_k \qquad \nu = \frac{1}{2\pi}\int d^2k\, B_k = \pm 1.$$

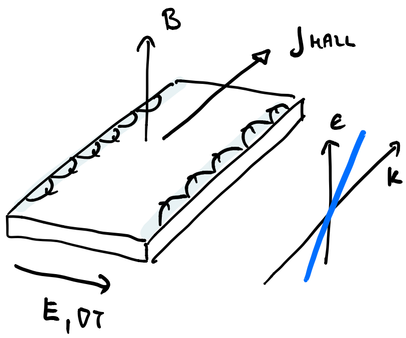
$$B_k = (\vec{\nabla}_k \times \vec{A}_k)_z$$

$\Longrightarrow$ Edge modes.

$$\boxed{\nu_{edge} = (\#\text{ right movers} - \#\text{ left movers}) = \nu}$$

Quantized Thermal Hall Effect

Recall that in the IQHE, the presence of edge states led to a quantized Hall conductance.



$$\sigma_{xy} = \frac{e^2}{h}\nu \; ; \qquad \sigma_{xx} = 0$$

ν = # of edge states.

Now under rather general conditions

$$\frac{1}{T}K_{ab} = L\,\sigma_{ab}, \qquad L = \frac{\pi^2 k_B^2}{3e^2},$$

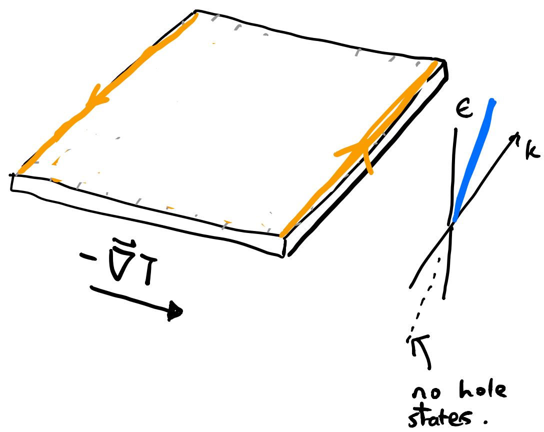
where K is the thermal conductivity tensor. So for the IQHE, we expect

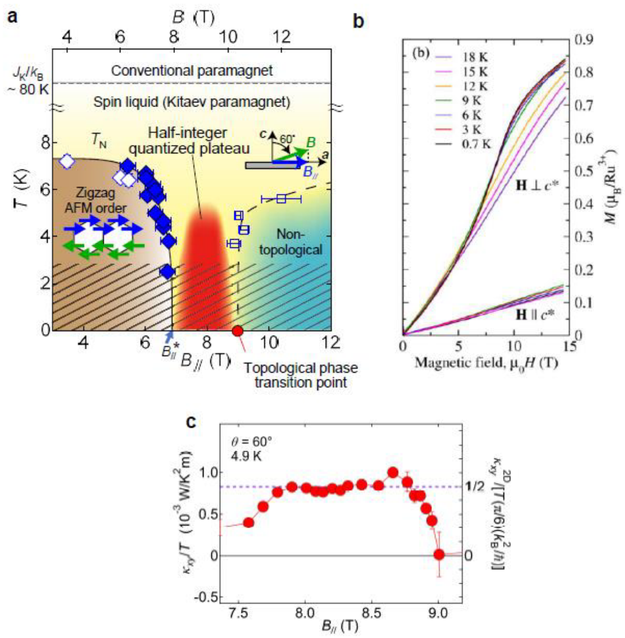
$$\frac{K_{xy}}{T} = \frac{\pi^2 k_B^2}{3e^2} \times \frac{e^2}{h}\nu = \left(\frac{\pi k_B^2}{6\hbar}\right)\nu$$

A Majorana edge state carries ½ the heat current of an electron edge state

$$\Rightarrow \boxed{\frac{\kappa_{xy}}{T} = \frac{1}{2}\frac{\pi k_B^2}{6\hbar}}$$

A PURE THERMAL HALL EFFECT.





Kasahara et al.

Nature, 559, 227-231 (2018)

NON ABELIAN ANYONS IN THE B-PHASE

Vortices now bind unpaired Majoranas.

1 - vacuum   σ - vortex   ε - fermion.

$$\varepsilon \times \varepsilon = 1, \quad \varepsilon \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \varepsilon$$

Anyons now have NON-ABELIAN braiding rules.

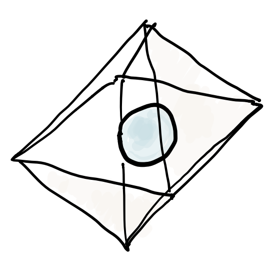
Kitaev Spin Liquids

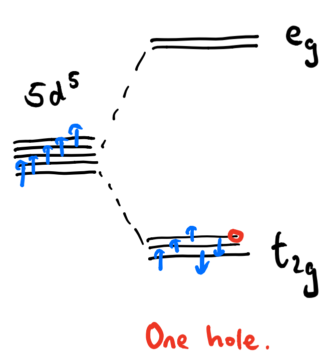
In 2009, George Jackeli & Giniyat Khaliulin proposed that Iridium atoms inside octahedra would develop Kitaev-like Ising interactions with their neighbors. Suddenly, the Kitaev honeycomb model was no longer a "toy": it might be realized in solid state quantum materials.

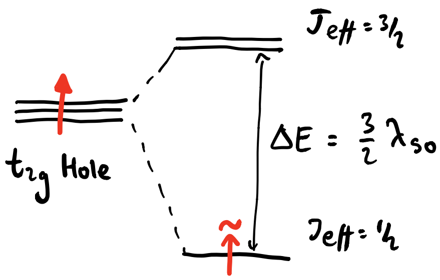
Proposed systems: spin orbit coupled transition metals (e.g Ru, Ir) in an Octahedral environment

Ir^{4+} $5d^5$

Ru $4d^5$.



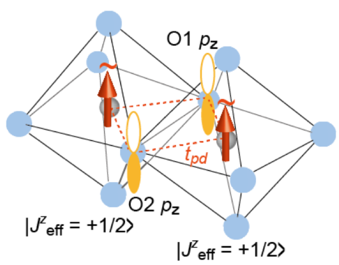




$$\sqrt{\tfrac{1}{3}}\,|\ell^z = 0, \uparrow\rangle + \sqrt{\tfrac{2}{3}}\,|\ell^z = +1, \downarrow\rangle$$

Spin orbit splits the t_{2g} orbitals into a low + high spin J-state.

"Mottness"

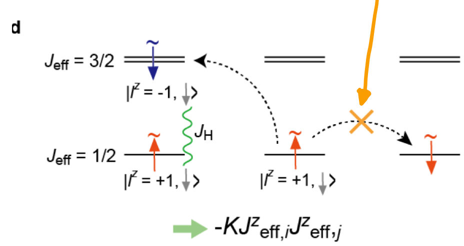


Edge sharing octahedra.

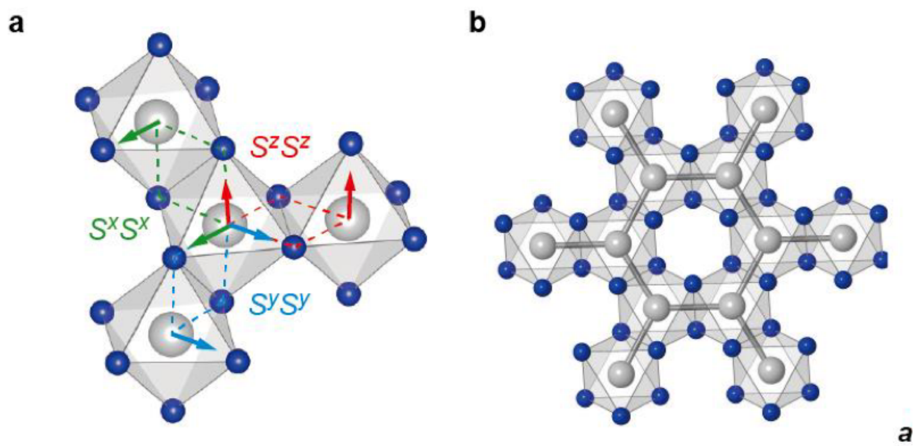
e.g IrO_6 or $RuCl_6$.

Suppressed by Q. Interference.

Hunds coupling LOWERS the energy of virtual hops of two parallel spins, inducing an Ising coupling perpendicular to the plane shared by the edge & the Ir atoms.



Octahedra can be laid down in a plane to form a honeycomb structure

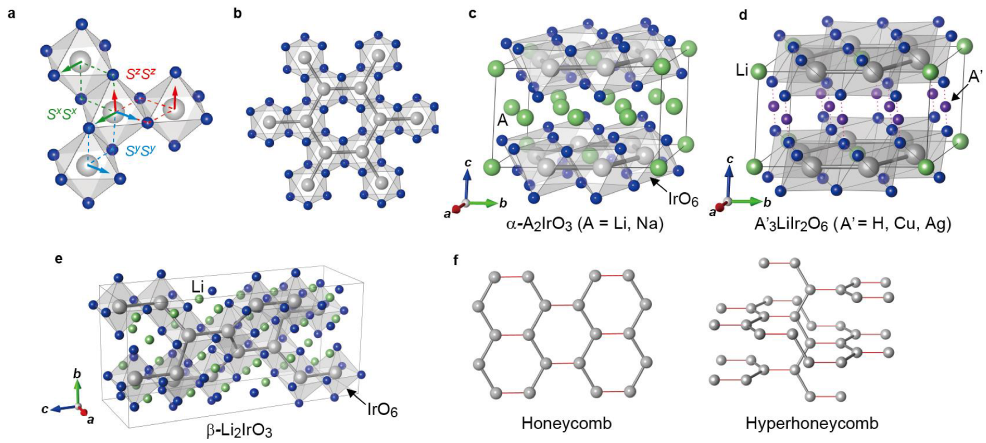


The three neighboring edge-sharing octahedra define three orthogonal planes, giving rise to the three directions for the Ising interactions. Interlayer couplings are weak, but there are additional Heisenberg & tensor interactions.

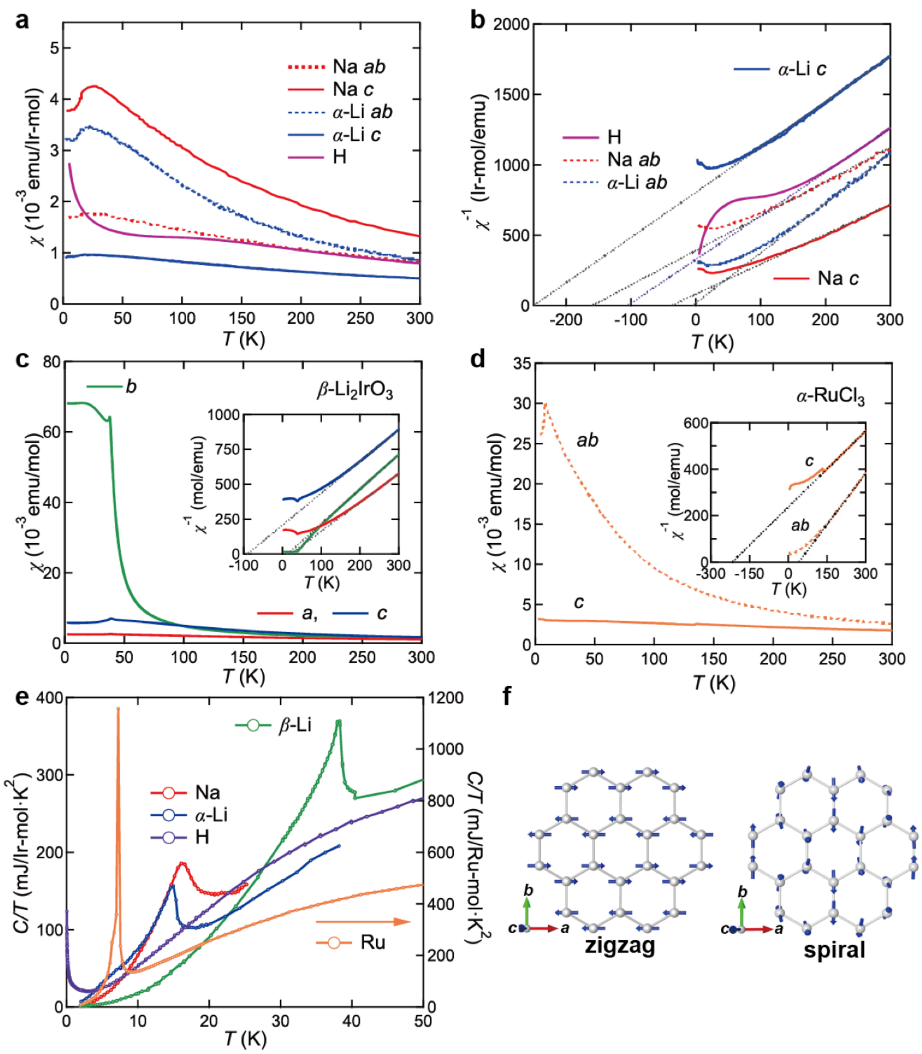
$$H = \sum_{\langle i,j \rangle_\gamma} \left\{ -K S_i^\gamma S_j^\gamma + \Gamma \left(S_i^\alpha S_j^\beta + S_i^\beta S_j^\alpha \right) + J \vec{S}_i \cdot \vec{S}_j \right\}$$

Hunds   d-d / d-p electron transfer   d-d hybridization

$(K > \Gamma \gg J)$.

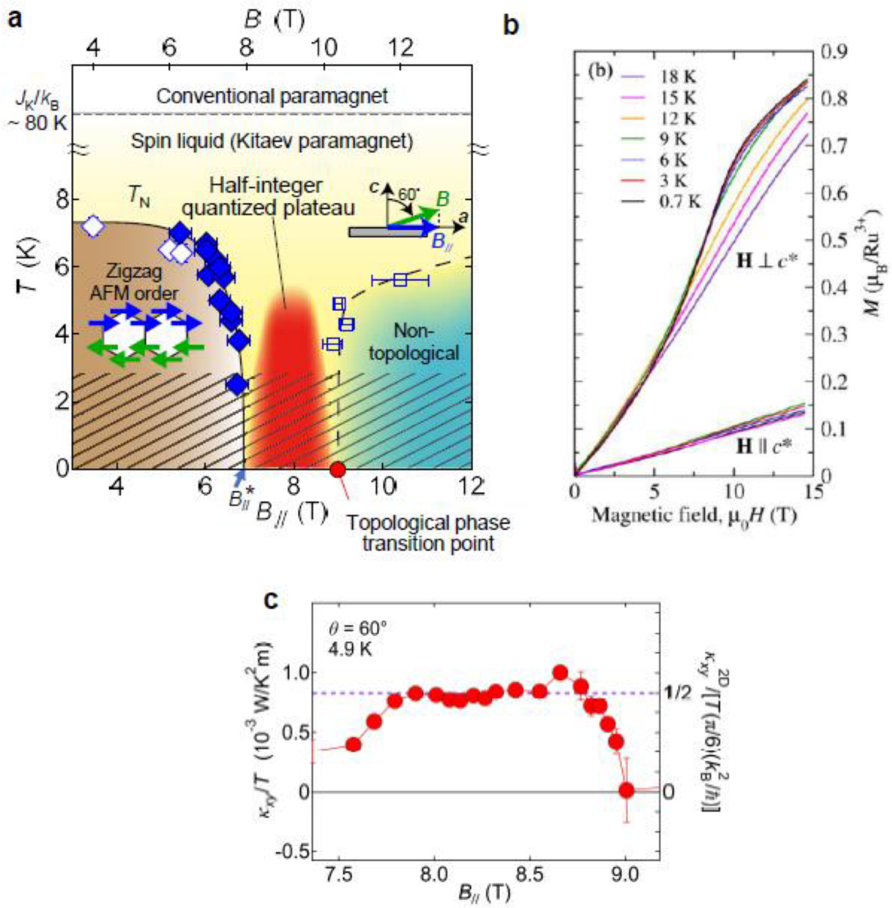


Materials	Crystal structure (Space group)	T_{mag}	anisotropy	p_{eff} (μ_B)	θ_{CW} (K)	Magnetic ground state	Ref.
Na_2IrO_3	2D (*C*2/*m*)	15 K	$\chi_c > \chi_{ab}$	1.81 (*ab*) 1.94 (*c*)	-176 (θ_{ab}) -40 (θ_c)	zigzag	40,57,66, 67
α-Li_2IrO_3	2D (*C*2/*m*)	15 K	$\chi_{ab} > \chi_c$	1.50 (*ab*) 1.58 (*c*)	+5 (θ_{ab}), -250 (θ_c)	Spiral	44,65,70
$H_3LiIr_2O_6$	2D (*C*2/*m*)	-	$\chi_{ab} > \chi_c$	1.60	-105	Spin-liquid	46
Cu_2IrO_3	2D (*C*2/*c*)	2.7 K	Not known	1.93(1)	-110	AF order or Spin-glass	42
$Cu_3LiIr_2O_6$	2D (*C*2/*c*)	15 K	Not known	2.1(1)	-145	AF order	49
$Ag_3LiIr_2O_6$	2D (*R*-3*m**)	~12 K	Not known	1.77		AF order	48
α-$RuCl_3$	2D (*C*2/*m* or $P3_112$,or *R*-3) ; *T* and sample dependent	7 K and/or, 14 K See text	$\chi_{ab} > \chi_c$	2.33 (*ab*), 2.71 (*c*)	+39.6(θ_{ab}), -216.4 (θ_c)	zigzag	51,64,68, 69, 131
β-Li_2IrO_3	3D (*Fddd*)	38 K	$\chi_b > \chi_c > \chi_a$	1.87 (*a*) 1.80 (*b*) 1.97 (*c*)	-90.2 (θ_a) +12.9 (θ_b) +21.6 (θ_c)	Spiral	52,71,92
γ-Li_2IrO_3	3D (*Cccm*)	39.5 K	$\chi_b > \chi_c > \chi_a$	~1.6	+40	Spiral	53,72

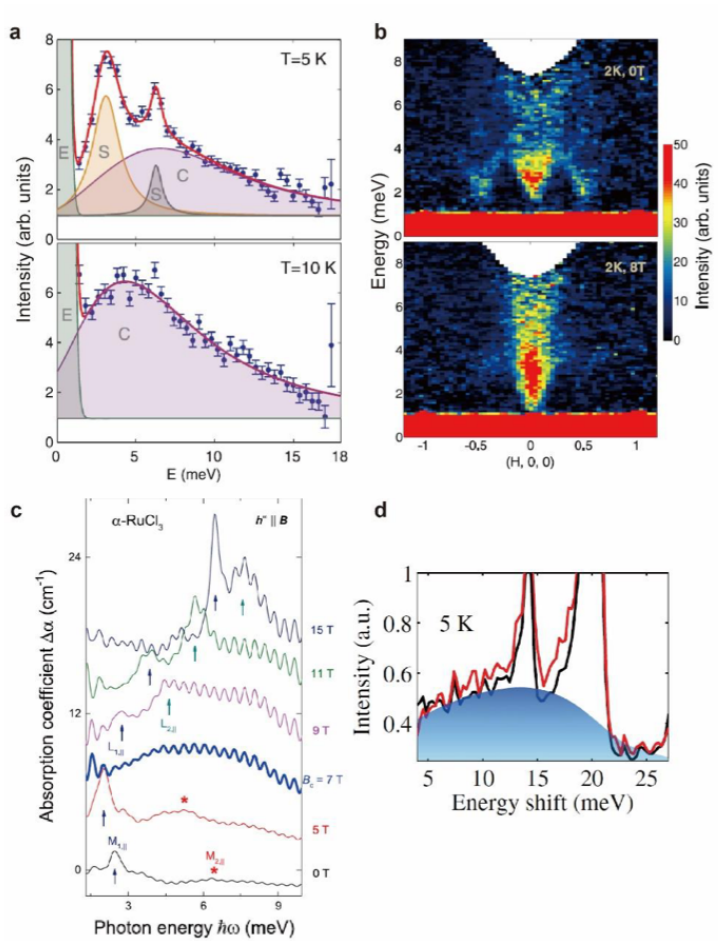


a
Na ab
Na c
α-Li ab
α-Li c
H
χ (10^-3 emu/Ir-mol)
T (K)
b
α-Li c
H
Na ab
α-Li ab
Na c
χ^-1 (Ir-mol/emu)
T (K)
c
b
β-Li2IrO3
χ^-1 (mol/emu)
T (K)
a, c
χ (10^-3 emu/mol)
T (K)
d
α-RuCl3
ab
c
χ^-1 (mol/emu)
T (K)
χ (10^-3 emu/mol)
T (K)
e
β-Li
Na
α-Li
H
Ru
C/T (mJ/Ir-mol·K^2)
C/T (mJ/Ru-mol·K^2)
T (K)
f
zigzag
spiral

Half Quantized Thermal Hall in RuCl$_3$



Signatures of Fractionalization Continuum.



Neutron

$S^+ \rightarrow$ Majoranas + Vortex Pair

Raman

$S.S \rightarrow$ Majoranas

Figure 7. **Signature of fractional excitations in α-RuCl$_3$. a,** Inelastic neutron scattering in single-crystal α-RuCl$_3$ measured at temperatures of T = 5 K (top) and 10 K (bottom)[75]. The data is integrated over a small reciprocal space volume centered at the Γ point of the two-dimensional lattice. The letters designate the contributions from the elastic line "E", spin-waves "S", and continuum scattering "C". **b,** Inelastic neutron scattering is measured at T = 2 K (top) in zero external magnetic field and (bottom) in a field of 8 T in the honeycomb plane, large enough to suppress the magnetic order[90]. The color bar denotes the relative intensity. **c,** THz spectroscopy measurements in α-RuCl$_3$[113] in the presence of a magnetic field applied in the honeycomb plane, with the THz field parallel to the applied field direction. All measurements were carried out at T = 2.4 K. The arrows indicate locations of excitations inferred from the data. **d,** Detail of Raman measurements in α-RuCl$_3$ at T = 5 K[107]. The blue shaded area represents the magnetic continuum scattering. (Panel **a** reproduced with permission from Ref. 75, panel **b** reproduced with permission from Ref. 90, panel **c** reproduced with mission from Ref. 113, and panel **d** reproduced with permission from Ref. 107).