Exercises . Physics 603. Kitaev Spin Liquid (Due Dec 4)

1. The Kitaev honeycomb model can, surprisingly be solved by using an old-fashioned Jordan Wigner transformation. (Jordan, who together with Klein was the inventor of fermion creation and annihilation operators.) To see how this works, first see that we can compress the honeycomb lattice in one direction and redraw it as a brick-wall lattice, composed of one dimensional chains with alternating cross-links (see Fig. 1), where the chains are labelled by the index l , and the position along the chains is labelled by the index j. so that the Hamiltonian can be written

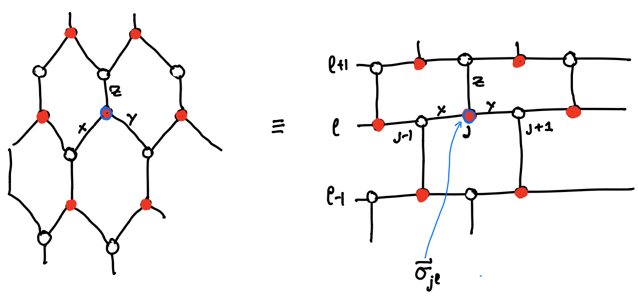

FIG. 1: Equivalence between honeycomb and "brick-wall" lattice.

$$H = -\sum_{l+j \in \text{even}} \left[J^x(\sigma^x_{jl}\sigma^x_{j-1,l}) + J^y(\sigma^y_{jl}\sigma^y_{j+1,l}) + J^z(\sigma^z_{jl}\sigma^z_{j,l+1}) \right] \tag{1}$$

(a) Show that the Jordan Wigner transformation can be generalized to each of the horizontal chains through the substitution:

$$\begin{aligned} \sigma^+_{jl} &= (\sigma^x_{jl} + i\sigma^y_{jl})/2 = f^\dagger_{jl} \exp\left[i\pi \sum_{r<j} n_r \right] \\ \sigma^-_{jl} &= (\sigma^x_{jl} - i\sigma^y_{jl})/2 = f_{jl} \exp\left[-i\pi \sum_{r<j} n_r \right] \\ \sigma^z_{jl} &= 2n_{jl} - 1, \end{aligned} \tag{2}$$

where $n_r = \sum_l f^\dagger_{rl} f_{rl}$ is the sum over all occupancies in the rth row of spins and f_{rl} is a fermion operator at site (r, l). Verify that the spin operators on different chains commute.

(b) Show that the fermionized Hamiltonian can be written in the form

$$H = -\sum_{l+j\in\text{even}} \Big[J^x (f_{jl}+f^\dagger_{jl})(f_{j-1l}-f^\dagger_{j-1l}) + J^y (f^\dagger_{j+1l}-f_{j+1l})(f^\dagger_{jl}+f_{jl}) + J^z (2n_{jl}-1)(2n^z_{j,l+1}-1) \Big]$$

(c) Split the fermions into their Majorana components, writing $f_{jl} = (c_{jl} + ib_{jl})/2$ ($j + l$ even) and $f_{jl} = (b_{jl} - ic_{jl})/2$ ($j + l$ odd), (where $c^2_{jl} = 1$) to show that

$$H = \frac{1}{2} \sum_{j+l\in\text{even}} [i(J^x c_{j-1l} + J^y c_{j+1l} + J^z c_{jl+1} u_{jl+1,jl}) c_{jl} + \text{H.c}] \tag{3}$$

where $u_{jl+1,jl} = -ib_{jl+1}b_{jl}$ is a Z_2 field that lives on the vertical z bonds. You should show that $\hat{u}_{jl+1,jl} = \pm 1$ commutes with the Hamiltonian.

(d) What is the relationship between this model, and the Kitaev model, written in terms of Majorana fermions?

(e) What is the spectrum of Majorana excitations in the ground-state?

(f) What excitation (s) is/are created by flipping the sign of $u_{jl+1,jl}$ and what is the approximate energy of the resulting state?