L9. RELATIVISTIC KINEMATICS

If we need to revise our notions of space and time to preserve the principle of relativity, then we also need to modify our understanding of the way objects move. This leads us to the subject of relativistic kinematics.

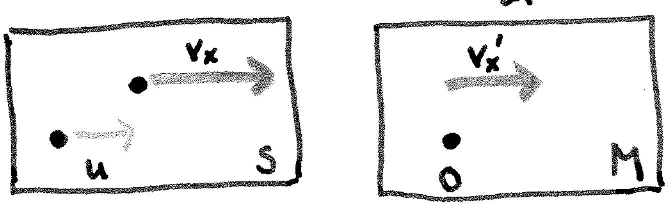
Today we will see how our understanding of velocity, momentum & energy are modified by relativity.

37.5b Relativistic Velocity Transformation

If we differentiate the Lorentz transformation, then

$$dx' = \gamma\,(dx - u\,dt)$$
$$dt' = \gamma\left(dt - \frac{u}{c^2}\,dx\right)$$

$$\Rightarrow \quad \frac{dx'}{dt'} = \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2}\frac{dx}{dt}}$$



$$\boxed{v_x' = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}}$$

Velocity Addition.

- If v_x & u are much smaller than c $\quad v_x, u \ll c$, then

  $v_x' = v_x - u$

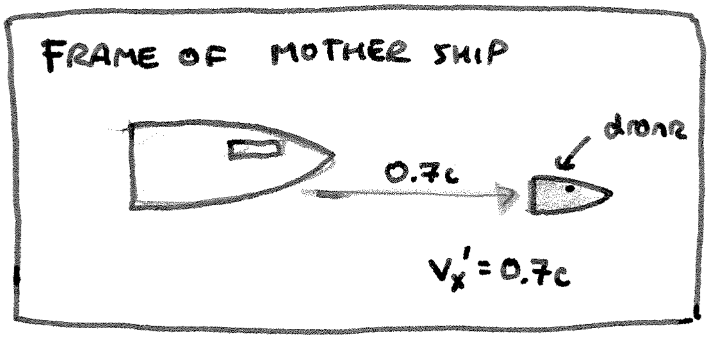
- If $v_x = c$, then

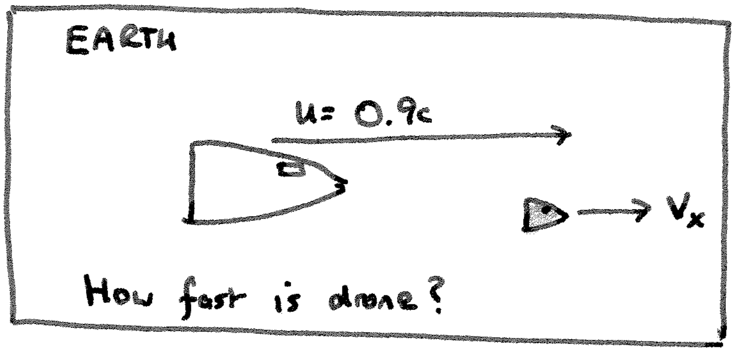
  $$v_x' = \frac{c - u}{1 - \frac{cu}{c^2}} = \frac{c-u}{1 - u/c} = c$$

- If we change $u \to -u$, $v_x \leftrightarrow v_x'$ or solve for v_x

  $$\boxed{v_x = \frac{v_x' + u}{1 + \frac{u v_x'}{c^2}}}$$

37.5b Example

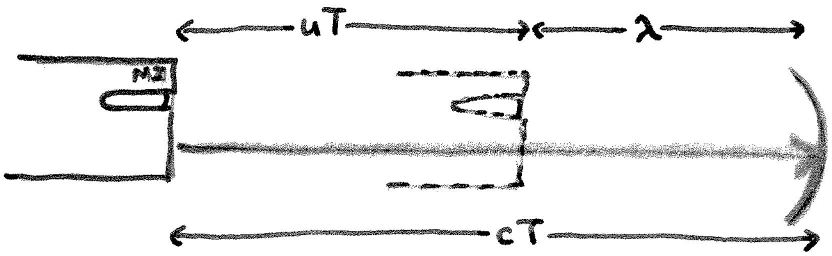




$$V_x = \frac{V_x' + u}{1 + \frac{V_x' u}{c^2}} = \frac{0.7c + 0.9c}{1 + \frac{(0.7c)(0.9c)}{c^2}} = 0.982c.$$

37.6 DOPPLER EFFECT FOR E.M WAVES

When a source of sound moves towards us, it acquires a higher pitch. When it moves away from us, it has a lower pitch. This is called the Doppler effect. The same effect also occurs for electromagnetic waves, but now we must take into account the effect of time dilation.



$$\lambda = (c-u)\,T$$

$$f = \frac{c}{\lambda} = \left(\frac{c}{c-u}\right)\frac{1}{T} \qquad (1)$$

But $T = \dfrac{T_0}{\sqrt{1-\frac{u^2}{c^2}}}$ $\qquad \dfrac{1}{T} = \dfrac{1}{T_0}\sqrt{1-\frac{u^2}{c^2}} = f_0\sqrt{1-\frac{u^2}{c^2}} \qquad (2)$

Combining (1) & 2

$$f = \frac{c}{c-u}\sqrt{\frac{c^2-u^2}{c^2}}\, f_0$$

$$= \sqrt{\frac{c^2-u^2}{(c-u)^2}}\, f_0$$

$$= \sqrt{\frac{\cancel{(c-u)}(c+u)}{(c-u)^{\cancel{2}}}}\, f_0$$

$$\boxed{f = \sqrt{\frac{c+u}{c-u}}\, f_0}$$

E.M. DOPPLER

MOVING TOWARDS YOU	MOVING AWAY FROM YOU

e.g. Blue light of wavelength $\lambda = 400\text{nm}$ is emitted from a moving source. An observer measures its wavelength to be $\lambda' = 800\text{nm}$. How fast is the source moving relative to the observer?

$$f_0 = c/\lambda_0 = \frac{3 \times 10^8}{4 \times 10^{-}} = 0.75 \times 10^{15} = 750\,\text{THz}$$

$$f' = 1/2\, f_0 = \sqrt{\frac{c+u}{c-u}}\, f_0$$

$$\frac{1}{2} = \sqrt{\frac{c+u}{c-u}} \qquad \frac{1}{4} = \frac{c+u}{c-u}$$

$$\Rightarrow (c-u) = 4(c+u)$$

$$-3c = 5u \Rightarrow \boxed{u = -0.6c}$$

Source is moving away from observer @ 0.6 c.

37.7 RELATIVISTIC MOMENTUM

In a collision, the total mass & the total momentum are conserved. If this is to remain true for all inertial observers, we need to revise what we mean by momentum. It turns out that the form that guarantees momentum conservation holds for all inertial observers is

$$\boxed{\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma m_0 \vec{u}}$$

where m_0 is the (rest) mass of the particle.

Newton's law of motion now becomes

$$\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

If acceleration is in the same direction as motion

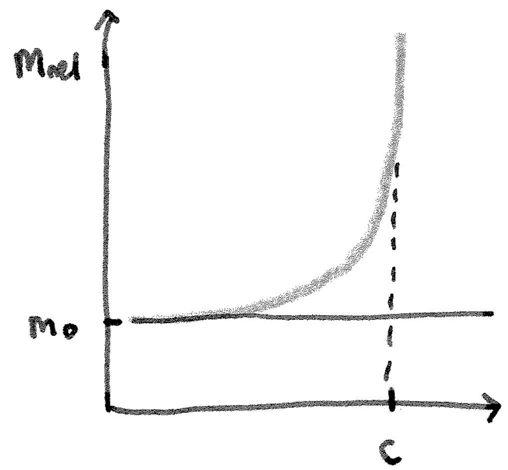
$$F = \frac{m_0 a}{\sqrt{1-u^2/c^2}} + \frac{m_0 u}{(1-u^2/c^2)^{3/2}} \frac{ua}{c^2}$$

$$= \frac{m_0 a}{(1-u^2/c^2)^{3/2}} = \gamma^3 m_0 a$$

$$a = \frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{3/2}$$

$$\boxed{\begin{aligned} p &= \gamma m_0 v \\ F &= \gamma^3 m_0 a \end{aligned}}$$

$$m_{rel} = \frac{m_0}{\sqrt{1-v^2/c^2}}$$



$$\boxed{\vec{p} = m_{rel}\,\vec{u}}$$

But note $F \neq m_{rel}\, a$, $K.E \neq \frac{1}{2} m_{rel} v^2$

Motion in a circle , γ = constant. $\vec{F} \perp \vec{v}$

$$\vec{F} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d\vec{v}}{dt} = m_{rel}\,\vec{a}$$

<u>ONLY</u> if $\vec{F} \perp \vec{v}$

In general $\vec{F}$ & $\vec{a}$ are <u>not</u> parallel.

e.g. Calculate momentum & accn. of e^- moving at $0.9c$ in an electric field of 5×10^5 N/C

$$F = qE = (1.6 \times 10^{-19} C) \times 5 \times 10^5 N/C = 8 \times 10^{-14} N$$

$$\gamma = \frac{1}{\sqrt{1-(0.9)^2}} = 2.29$$

$$p = \gamma m_o u = 2.29 \times (9.11 \times 10^{-31} kg) \times 0.9 \times 3 \times 10^8 m/s$$

$$= 5.6 \times 10^{-22} kg\,m/s^2$$

$$a = \frac{F}{\gamma^3 m_o} = \frac{8 \times 10^{-14}}{(2.29)^3 (9.11 \times 10^{-31})} = 7.3 \times 10^{15} m/s^2$$

37.8 RELATIVISTIC WORK & ENERGY

How much work do we do in accelerating a relavishi particle?

Classically, the work done is the change in kinetic energy, where $K.E = \frac{1}{2} m v^2$. What is it relativistically?

$$W = \int F dx = \int \frac{m_0 a}{(1 - u^2/c^2)^{3/2}} dx \qquad (\vec{F} \parallel \vec{u})$$

Now $a\,dx = \frac{du}{dt} dx = du\left(\frac{dx}{dt}\right) = du\,u = \frac{1}{2} d u^2$. So

$$W = \frac{m_0}{2} \int_{u_i}^{u_f} \frac{du^2}{(1 - u^2/c^2)^{3/2}}$$

$u^2/c^2 = x \Rightarrow dx = \frac{1}{c^2} du^2$

$$W = \frac{m_0 c^2}{2} \int \frac{dx}{(1-x)^{3/2}} = m_0 c^2 \left[\frac{1}{\sqrt{1 - u_f^2/c^2}} - \frac{1}{\sqrt{1 - u_i^2/c^2}} \right]$$

$$= E_f - E_i$$

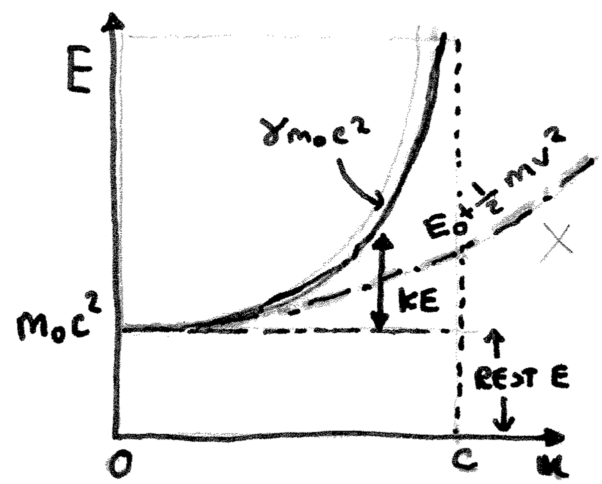
where

$$\boxed{E(u) = m_0 c^2 \frac{1}{\sqrt{1-u^2/c^2}}}$$

We may interprete

$$\boxed{E_0 = m_0 c^2}$$ Energy-mass relation.

as the rest energy. For small u



$$E(u) = m_0 c^2 \left[\left(1 - u^2/c^2\right)^{-1/2} \right]$$

$$= m_0 c^2 \left[1 + \frac{1}{2}\left(u^2/c^2\right) + \frac{3}{8}\left(u^4/c^4\right) + \dots \right]$$

$$\boxed{E(u) = \underset{\text{REST ENERGY}}{m_0 c^2} + \underset{\text{CLASSICAL K.E}}{\frac{1}{2} m_0 u^2} + \underset{\text{FIRST RELATIVISTIC CORRECTION.}}{\frac{3}{8} m_0 u^4/c^2} + \dots}$$

a) What is the rest energy of an electron in electron volts?

$$m_o c^2 = 9.109 \times 10^{-31} \times (2.998 \times 10^8)^2$$

$$= 8.187 \times 10^{-14} \text{ J}$$

$$m_o c^2 = eV \qquad V = \frac{m_o c^2}{e} = \frac{8.187 \times 10^{-14}}{1.602 \times 10^{-19}}$$

$$= 5.11 \times 10^5 \text{ eV}$$

$$= \underline{0.511 \text{ MeV}}$$

b) How fast is an e^- moving after accelerating through a potential of 5 MeV?

$$m_o c^2 (\gamma - 1) = 5 \times 10^6 e$$

$$\gamma - 1 = 5 \times 10^6 \left(\frac{e}{m_o c^2}\right) = \frac{5}{0.511} = 9.78$$

$$\gamma = 10.78 \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \Rightarrow \gamma^2 = \frac{1}{1 - u^2/c^2} \Rightarrow \frac{u}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$\frac{u}{c} = \sqrt{1 - \frac{1}{10.78^2}} \simeq 0.996 \qquad \boxed{u = 0.996\,c}$$