Transforming Polar to Cartesian and Vice Versa

In discussion section I mentioned that once we specify an origin, we have a one-to-one correspondance between vectors and points. Given a point, we just associate the vector which starts at the origin and points to that point. Going the other way: Given a vector, we draw it starting at the origin - where it is ends is a point, and we associate that point to the vector.

What this means is that talking about points is just like talking about vectors, as long as we agree to work in a specified coordiante system with an origin. With this in mind, let's start by writing down the rules for converting between Cartesian and Polar coordinates in the plane.

For a point given by Cartesian corordiantes, $(x, y)_{\text{Cart}}$, we need to specify the coordinates in Polar form in terms of the Cartesian data x and y. This is easy to do once you draw the point and a right triangle. The polar length is obtained with the pythagorean theorem, while the angle is obtained by an application of the inverse tangent. The answer is:

$$(r, \theta)_{\text{Polar}} = (\ \sqrt{x^2 + y^2}\ , \arctan \frac{y}{x}\ )_{\text{Polar}}$$

Meanwhile, for a point given by Polar coordinates, $(r, \theta)_{\text{Polar}}$, we need to specify the coordinates in Cartesian form in terms of the Polar data r and θ. We can again draw a right triangle. Using the sine and cosine functions, and a bit of algebra, we get the anwser:

$$(x, y)_{\text{Cart}} = (\ r\cos(\theta)\ ,\ r\sin(\theta)\ )_{\text{Cart}}$$

For now on we will work with vectors specified by points, which themselves will be specified in either Cartesean or Polar coordinates.

Vector Addition

For two vectors given by points in Cartesean coordinates:

$$\overrightarrow{u} = (u_x, u_y)_{\text{Cart}}$$

$$\overrightarrow{v} = (v_x, v_y)_{\text{Cart}}$$

The sum is

$$\overrightarrow{u} + \overrightarrow{v} = (u_x + v_x, u_y + v_y)_{\text{Cart}}$$

In other words, you just sum component by component.

For two vectors given by points in Polar coordiantes:

$$\overrightarrow{u} = (u_r, u_\theta)_{\text{Polar}}$$

$$\overrightarrow{v} = (v_r, v_\theta)_{\text{Polar}}$$

The sum in Polar coordinates is harder to obtain. The best strategy is probably to convert to Cartesian coordiantes, sum in those coordinates, and then covert back. This yields:

$$\overrightarrow{u} + \overrightarrow{v} = \left(\sqrt{u_r^2 + v_r^2 + u_r v_r (\cos(u_\theta)\cos(v_\theta) + \sin(u_\theta)\sin(v_\theta))}\,, \right.$$
$$\left. \arctan \frac{u_r \sin(u_\theta) + v_r \sin(v_\theta)}{u_r \cos(u_\theta) + v_r \cos(v_\theta)} \right)_{\text{Polar}}$$

Needless to say, you will probably not want to use Polar form when you are summing vectors frequently ...

Multiplication of a Vector by a Scalar

For a vector given by a point in Cartesian coordinates:

$$\overrightarrow{u} = (x, y)_{\text{Cart}}$$

The result when you multiply by the scalar a is simply

$$a\overrightarrow{u} = (ax, ay)_{\text{Cart}}$$

Just multiply each component by the scalar.

For a vector given by a point in Polar coordinates:

$$\overrightarrow{u} = (r, \theta)_{\text{Polar}}$$

The result when you multiply by the scalar a is even simpler:

$$a\overrightarrow{u} = (ar, \theta)_{\text{Polar}}$$

Direction unchanged, just scale the length.

Dot Product of Two Vectors

For two vectors given by points in Cartesean coordinates:

$$\overrightarrow{u} = (u_x, u_y)_{\text{Cart}}$$

$$\overrightarrow{v} = (v_x, v_y)_{\text{Cart}}$$

The dot product is

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_x v_x + u_y v_y$$

the cooresponding coordinates are multiplied, and then summed together.

For two vectors given by points in Polar coordiantes:

$$\overrightarrow{u} = (u_r, u_\theta)_{\text{Polar}}$$

$$\overrightarrow{v} = (v_r, v_\theta)_{\text{Polar}}$$

The dot product can be written as

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_r v_r \cos(\theta)$$

where θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. In two dimensions this will always be given by $|u_\theta - v_\theta|$ as long as you interpret angles above 180deg and above 360deg correctly. In three dimensions it is somewhat more difficult to compute the angle between vectors, and usually the best solution is simply to use 3-D Cartesian coordiantes instead.

Length of the Cross Product of Two Vectors

For completeness, I will list the length of the cross product. For two dimensions, the direction is always the same. Assuming the vectors are drawn on the page, the cross product points either 'out' or 'in' of the page, depending on which order the cross product is taken (i.e., depending if one forms $\overrightarrow{u} \times \overrightarrow{v}$ or $\overrightarrow{v} \times \overrightarrow{u}$).

For two vectors given by points in Cartesean coordinates:

$$\overrightarrow{u} = (u_x, u_y)_{\text{Cart}}$$

$$\overrightarrow{v} = (v_x, v_y)_{\text{Cart}}$$

The magnitude of the cross product is

$$|\overrightarrow{u} \times \overrightarrow{v}| = u_x v_y - u_y v_x$$

For two vectors given by points in Polar coordiantes:

$$\overrightarrow{u} = (u_r, u_\theta)_{\text{Polar}}$$

$$\overrightarrow{v} = (v_r, v_\theta)_{\text{Polar}}$$

The magnitude of the cross product is

$$|\overrightarrow{u} \times \overrightarrow{v}| = u_r v_r \sin(\theta)$$

where again θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.