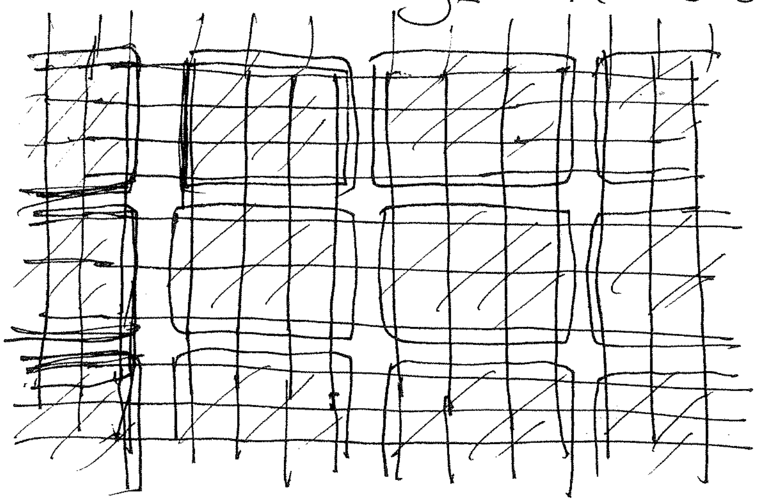
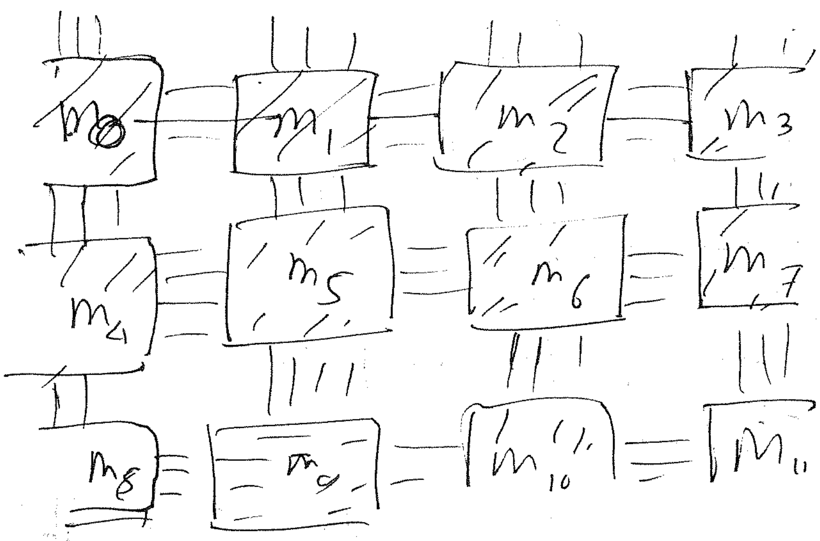
Renormalization Group



Stat mech problem! Sum over many degrees of freedom

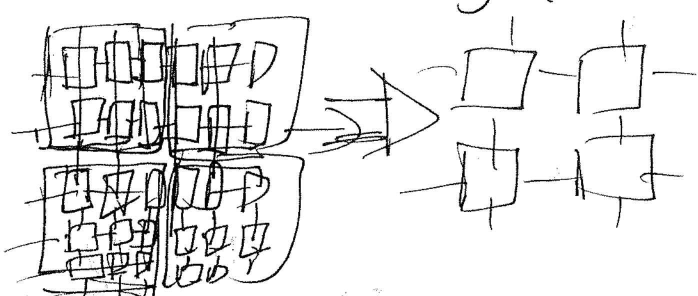
Landau



$$H(m(x), K, r, u, h_i;) = \int \left\{ \frac{1}{2} K (\nabla m(x))^2 - h m + \frac{1}{2} r\, m(x)^2 + \frac{u}{4} m(x)^4 + \dots \right\} dx$$

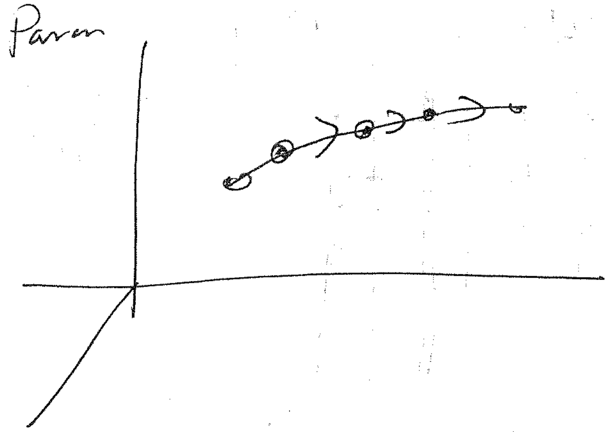
Landau then optimizes it to calculate various things + Gaussian approx around minima

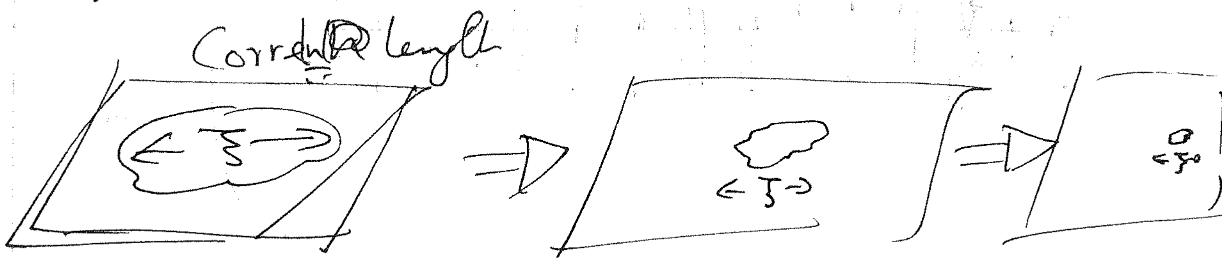
But the right way to do it is to keep on doing it



Eliminate more degrees of freedom map into another point in param space.. with a shrunken lattice

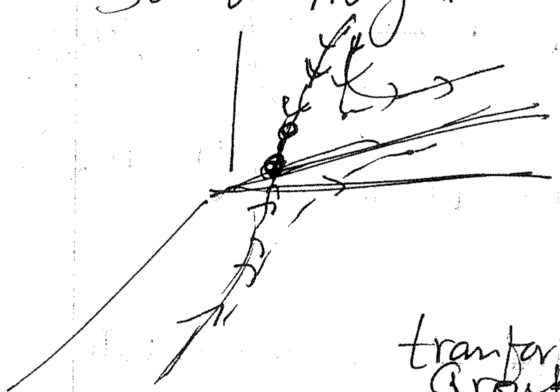
$$H(m(x), K, r, u, h) \Rightarrow H(m'(y), K', r', u', h', \dots)$$





At critical point $\xi \to \infty$

So we might converge



Param space

These sets of transformations have been called renormalization group. Some don't like the name.

Let us do this in practice by some simple form of RG (Renormalization group).

Eliminate half the degree of freedom at every stage: Decimation, one of many ways of doing RG in real space.

Example: 1D Ising

$\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ$

$$Z = \sum_{\{\sigma\}} e^{K(\cdots + \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_5 + \cdots)}$$

$$= \sum_{\cdots, \sigma_1, \sigma_3, \cdots} \ \sum_{\cdots \sigma_2, \sigma_4, \cdots} e^{K\{\cdots + \sigma_2(\sigma_1+\sigma_3) + \sigma_4(\sigma_3+\sigma_5) + \cdots\}}$$

$$= \sum_{\{\sigma_i\}_{i\,\text{odd}}} \cdots \left\{ e^{K(\sigma_1+\sigma_3)} + e^{-K(\sigma_1+\sigma_3)} \right\}$$

$$\left\{ e^{K(\sigma_3+\sigma_5)} + e^{-K(\sigma_3+\sigma_5)} \right\}$$

$$\cdots\cdots$$

$$= \sum_{\{\text{odd}\,\sigma_i\}} \prod_{i\,\text{odd}} 2\cosh K(\sigma_i + \sigma_{i+2})$$

~~For~~ $2\cosh K(\sigma+\sigma') = 2\cosh 2K$, when $\sigma = \sigma'$

$= 2$, when $\sigma = -\sigma'$

$$2\,\mathrm{ch}\,K(\sigma+\sigma') = 2\,\mathrm{ch}^{1/2} 2K \,(\mathrm{ch}^{1/2} 2K)^{\sigma\sigma'}$$
$$= f(K)\, e^{K'\sigma\sigma'}$$

$$e^{K'} = \mathrm{ch}^{1/2} 2K$$

or $\boxed{K' = \frac{1}{2} \ln \mathrm{ch}\, 2K}$

$$f(K) = 2\,\mathrm{ch}^{1/2} 2K$$

$$Z_N(K) = f(K)^{N/2} Z_{N/2}(K')$$

Define $\zeta(K)$ $\left(= -\frac{F}{NK_BT}\right)$ by $\ln Z_N(K) \simeq N\zeta(K)$

$$\zeta(K) = \frac{1}{2}\ln f(K) + \frac{1}{2}\zeta(K')$$

or
$$\boxed{\begin{aligned} \zeta(K') &= 2\zeta(K) - \ln f(K) \\ &= 2\zeta(K) - \ln 2\,\mathrm{ch}^{1/2} 2K \end{aligned}}$$

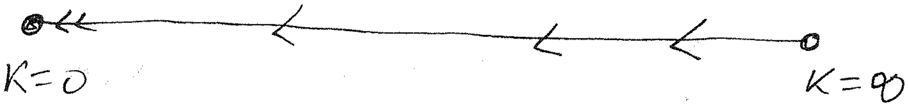
Now let us analyze the recursion in K.

For small K $\quad \mathrm{ch}2K = 1 + 2K^2 + \cdots$

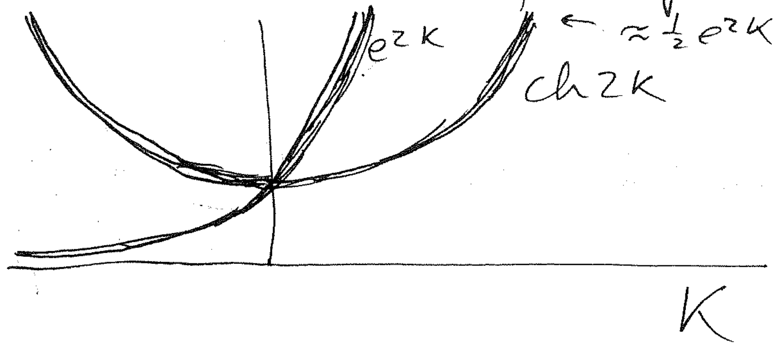
$$K' = \frac{1}{2} \ln \mathrm{ch}2K \approx \frac{1}{2} \ln(1 + 2K^2 + \cdots) \approx K^2$$

For large K. $\quad \mathrm{Ch}\, 2K \simeq \frac{1}{2} e^{2K}$

$$K' = \frac{1}{2} \ln \mathrm{ch}2K \approx \frac{1}{2}(2K - \ln 2) = K - \frac{1}{2} \ln 2$$



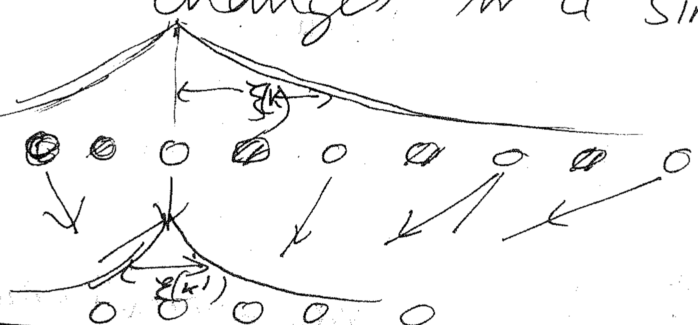
No intermediate fixed point



$$e^{2K'} = \mathrm{ch}2K$$

$K = 0$, the high temp fixed point, stable
$K = \infty$, " low " " ", unstable

Note that with decimation correlation length changes in a simple manner



$$\xi(K') = \frac{1}{2} \xi(K)$$

At low temp $e^{2K'} \approx \frac{1}{2} e^{2K}$ suggesting $\xi(K) \sim e^{2K}$. This is indeed true

$$\xi(K) = \frac{1}{\ln \operatorname{th} K} \approx \frac{1}{2} e^{2K}$$

For large K $\quad \zeta(K') = 2\zeta(K) - \ln 2 \operatorname{ch}^{1/2} 2K$

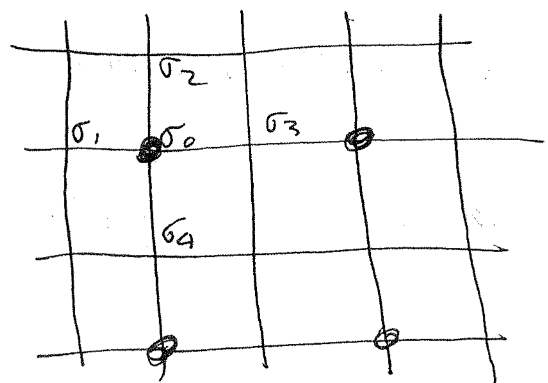
$$\Rightarrow \zeta\left(K - \tfrac{1}{2}\ln 2\right) \simeq 2\zeta(K) - \tfrac{1}{2}\ln 2 - K$$

Consistent with $\zeta(K) \sim K$

Since $Z_N \approx (2 \operatorname{ch} K)^N$ $\quad \zeta = \ln(2 \operatorname{ch} K) = \ln(e^K + e^{-K}) \approx K$ for large K

So we could figure out things about the unstable low temp fixed point from RG. Unfortunately we do not have a finite K transition.

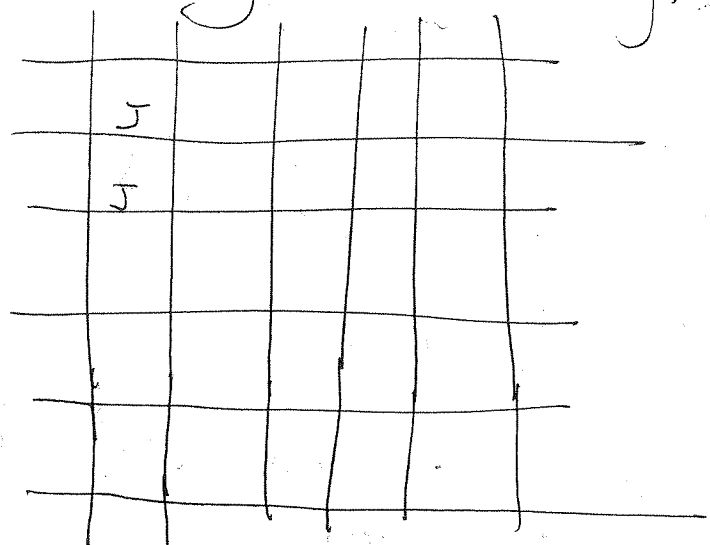
What about 2D Ising

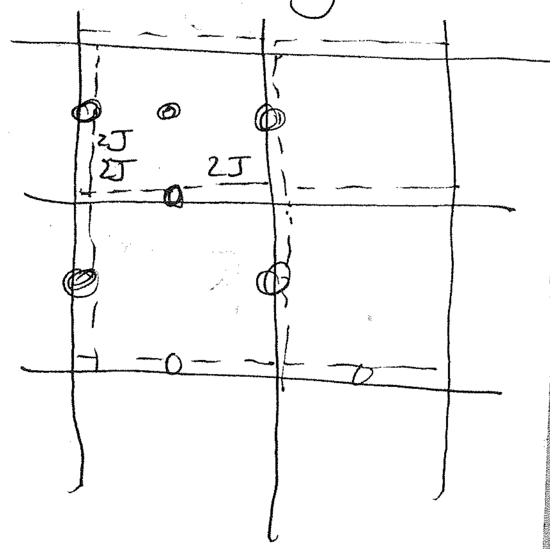


Trouble: each spin connected to four spins

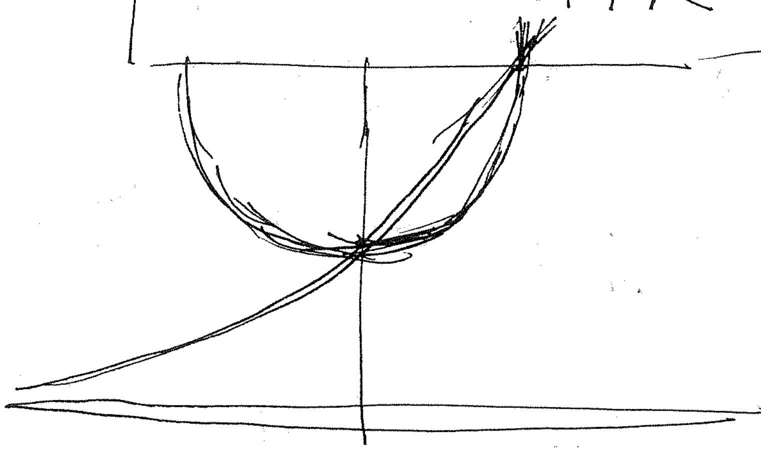
Need approximations!

Migdal Kadanoff bond moving



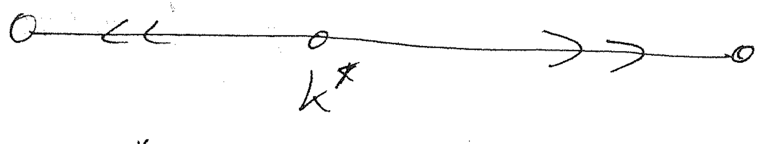


$$\boxed{e^{2K'} = \mathrm{ch}\, 4K}$$



$\mathrm{ch}\, 4K$
$\sim \frac{1}{2} e^{4K}$
larger than
e^{2K}

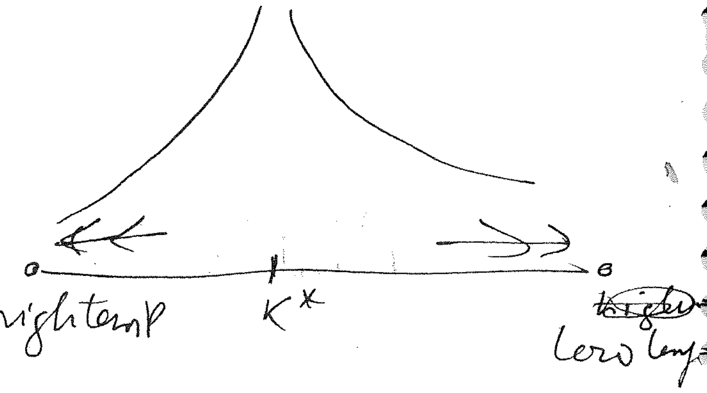
For small K $\qquad K' \simeq 4K^2$
For large K $\qquad K' \simeq 2K - \frac{1}{4}\ln 2$



$$e^{2K^*} = \mathrm{ch}\, 4K^* \qquad K^* = 0.30\,2\,\cancel{4}$$

MF $K^* = \beta J_c = \frac{1}{8} = 0.25$ $\qquad$ Exact $K_c = \frac{1}{2}\ln(1+\sqrt{2}) = 0.44\ldots$

How does ξ go as $(K-K^*)$



In each iteration $\xi(K') = \frac{1}{2}\xi(K)$

If $K - K^* = \delta K$ (or ε) is small

$$K' = \frac{1}{2}\ln ch(4K)$$

$$\Rightarrow \quad \delta K' = \frac{1}{2} \times 4 \times th(4K^*)\ \delta K$$

$$\delta K' = 2 \times 0.834 \quad \delta K$$

$$= 1.68\ \delta K$$

If $\quad \xi \sim \varepsilon^{-\nu} \sim (\delta K)^{-\nu}$

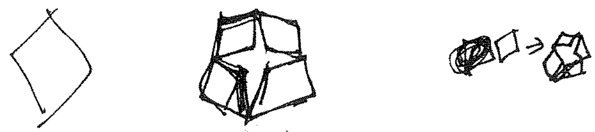
$$\frac{1}{2} = \frac{1}{1.68^{\nu}} \quad \text{or} \quad \nu = \frac{\ln 2}{\ln 1.68} = 1.34$$

MF $\nu = \frac{1}{2} = 0.5 \qquad$ Exact $\nu = 1$

Not great but not too bad either.

Problem: Somewhat arbitrary approximation.

(a) Decimation is exact on certain fractal lattices like hierachical latic



$$e^{2K'} = (\mathrm{ch}\, 2K)^2$$

or $K' = \frac{1}{2} \ln \mathrm{ch}\, 2K$

Once more a non trivial fixed point

(b) We could improve the recursion for 2D Ising

Instead of $K' = \frac{1}{2} \ln \mathrm{ch}\, 4K$

$$K' = \frac{3}{8} \ln \mathrm{ch}\, 4K$$

Does a much better job for K^*

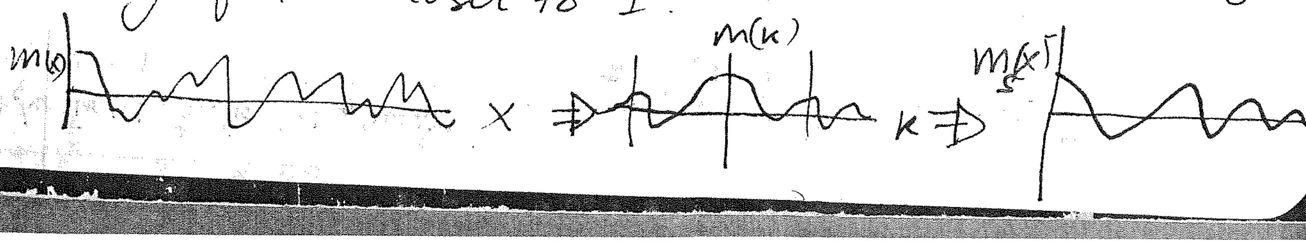
$K^* = 0.506$ (0.44 exact)

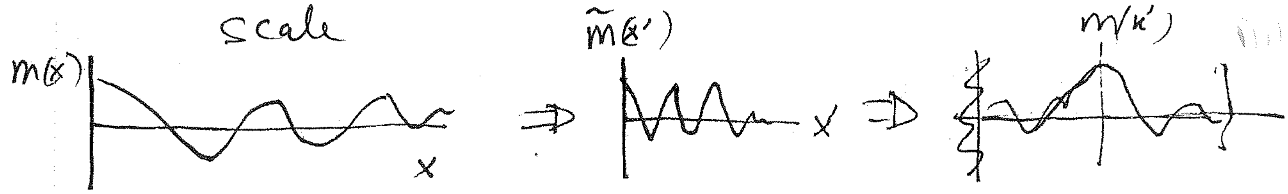
$\nu =$ (1 exact)

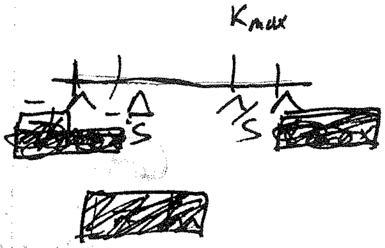
$\mathrm{th}\, 4K^* = \mathrm{th}\, 2.028 = 0.965$

$\frac{3}{2}\, \mathrm{th}\, 4K^* = 1.44$

Systematic RG requires us to consider momentum space [fourier modes] and scaling by factors closer to 1.







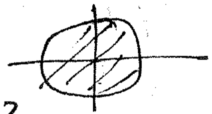
$$x = x's$$

$|k|<\Lambda_s$ $\quad \frac{\Lambda}{s}<|k|<\Lambda$

$$m(x) = m_s(x) + m_f(x)$$

slow $\qquad$ fast

Set $K=1$ by Scaling m

In higher D  kspace $|\vec{k}|<\Lambda$

$$\int d^Dx\left[\frac{1}{2}(\vec{\nabla} m)^2 + \frac{r}{2} m^2\right] = \int d^Dx\left(\frac{1}{2}(\nabla m_s)^2 + \frac{r}{2} m_s^2\right) + \int d^Dx\left(\frac{1}{2}(\nabla m_f)^2 + \frac{r}{2} m_f^2\right)$$

Crossterms dissapear

$$\int_x m_s m_f = \int_k\int_{k'} \hat{m}_s(k) e^{ikx} e^{+ik'x} \hat{m}_f(k') = 2\pi\int_{k,k'} \delta(k+k')\hat{m}_s(k)\hat{m}_f(k')$$
$$= 0$$

Same for $\int_x \nabla m_s \nabla m_s$

$$\int d^Dx\ \frac{1}{2}\left((\nabla m_s)^2 + r m_s^2\right) = \frac{s^D}{2}\int d^Dx'\left[\frac{1}{s^2}(\nabla' m_s)^2 + r m_s^2\right]$$

Setting $K=1$ $\quad s^{\frac{D-2}{2}} m_s = m' \qquad m_s = s^{\frac{2-D}{2}} m'$

$$\frac{1}{2}\int d^Dx'\left[(\nabla' m')^2 + s^2 r\, m'^2\right]$$

r grows. Remember $\xi \sim \frac{1}{\sqrt{r}}$ in MF + fluct. $\xi \sim \frac{1}{s}\xi$ as it should be

$$\frac{u}{4}\int d^Dx\, m^4 = \frac{u}{4}\int d^Dx\left[m_s^4 + \text{whole bunch of crossterms}\right]$$
$$4m_s^4 m_f + 6m_s^2 m_f^2 + 4m_s m_f^3 + m_f^4$$

Take only $\frac{u}{4}\int d^Dx\, m_s^4 \to \frac{u}{4} s^D \int d^D x'\, s^{-2D+4} m'^4$

$$= \frac{u s^{4-D}}{4}\int d^Dx'\, m'^4$$

Apart from the effect of crossterms effect of u grows for $D<4$
u can't be ignored. It grows slowly for $\epsilon = 4-D$ small

Small ϵ expansion: Of course we will like to know about 3 D, i.e. $\epsilon = 1$

Effect of crossterms $\frac{3u}{2}\int m_s^2 m_f^2$, av. over m_f

$$\frac{3u}{2}\int m_s^2(x)\langle m_f^2(x)\rangle d^Dx$$

$$= \frac{3u}{2} G_f(0)\int m_s^2\, d^Dx \quad \text{modifies } r$$

$$\frac{1}{2}\left(\frac{u}{4}\int d^Dx\, m_s^2 m_f^2\right)^2 \quad \text{av over}$$

$$-36\times\frac{1}{2}\frac{u^2}{4^2}\int d^Dy\int d^Dx\, m_s^2(x)\langle m_f^2(x)\, m_f^2(y)\rangle_c\, m_s^2(y) = -36\times\frac{1}{2}\times\frac{u^2}{4}\times 2\int m_s^4\langle m_f(x) m_f(y)\rangle^2$$

$$\approx -\frac{9}{4}u^2\left(\int d^Dz\, G_f^2(z)\right)\int d^Dx\, m_s^4(x)$$

$z = x - y$

where $G_f(\vec z) = \langle m_f(\vec x)\, m_f(\vec x+\vec z)\rangle$

$$m_f(\vec x) = \sum_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{1}{\sqrt{V}}\hat m_k e^{i\vec k\cdot\vec x} \Rightarrow e^{-\int \frac{1}{2}((\nabla m_f)^2 + r m^2)\, d^Dx} = \prod_{\substack{k\\ \frac{\Lambda}{s}<|k|<\Lambda}} e^{-\sum_k \frac{1}{2}(k^2+r) m_k^2}$$

$A = \{k \mid \frac{\Lambda}{s}<|k|<\Lambda\}$

$$\langle m_k^2\rangle = \frac{1}{k^2+r}$$

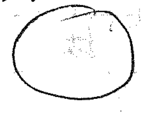
$$G_f(\vec z) = \frac{1}{V}\sum_{\frac{\Lambda}{s}<|k|<\Lambda}\frac{e^{i\vec k\cdot\vec z}}{k^2+r} \qquad G_f(0) = \frac{1}{V}\sum_{k\in A}\frac{1}{k^2+r} \approx \frac{1}{(2\pi)^D}\int_A d^Dk\,\frac{1}{k^2+r}$$

$$\int d^Dz\, G_f^2(z) = \frac{1}{V}\sum_{k\in A}\frac{1}{(k^2+r)^2} = \frac{1}{(2\pi)^D}\int_A d^Dk\,\frac{1}{(k^2+r)^2}$$

For $s = 1+t$ $\quad t$ small

$$\int_A d^D k\, f(|k|) \approx \Omega_D \Lambda^{D-1} (t\Lambda) f(\Lambda) \approx \Omega_D \Lambda^D t\, f(\Lambda)$$

Ω_D area or D dim sphere



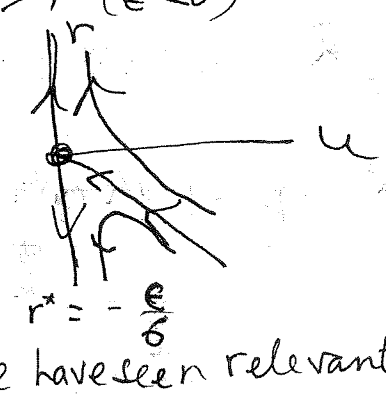
$$r' = \left(r + 3\frac{\Omega_D \Lambda^D}{(2\pi)^D} \frac{u t}{\Lambda^2 + r} \right)(1+t)^2$$

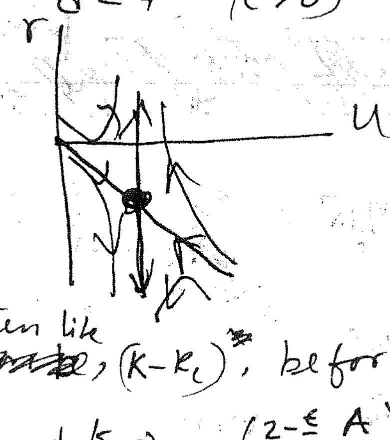
$$u' = \left(u - 9\frac{\Omega_D \Lambda^D}{(2\pi)^D} \frac{u^2 t}{(\Lambda^2 + r)^2} \right)(1+t)^{\epsilon}$$

Let us defined $A = \frac{3\Omega_D}{(2\pi)^D}$. For a parameter g, let $\frac{dg}{ds} = \frac{g'-g}{t}$ for small t. We work at $\Lambda = 1$ (eq 4.18 $r \to \Lambda^2 r$, $u \to \Lambda^{4-D} u$)

$$\frac{dr}{ds} = 2r + \frac{Au}{1+r}, \quad \frac{du}{ds} = \epsilon u - \frac{3Au^2}{(1+r)^2}$$

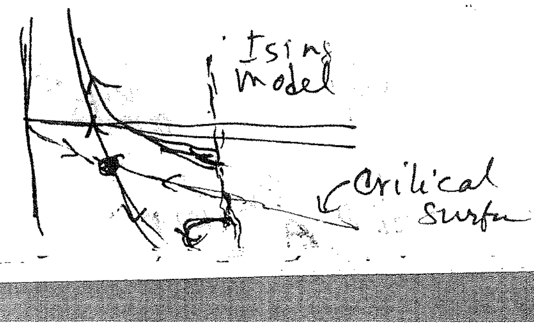
$d>4$ $(\epsilon<0)$ $\qquad$ $d<4$ $(\epsilon>0)$





Non triv FP: $u^* = \frac{\epsilon}{3A}$ $\quad r^* = -\frac{\epsilon}{6}$

We have seen relevant parameters like $(K-K_c)$, before



$$\frac{d}{ds}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2-\frac{\epsilon}{3} & A \\ 0 & -\epsilon \end{pmatrix}$$

$\lambda_\epsilon = 2 - \frac{\epsilon}{3} + \ldots \quad \nu = \frac{1}{\lambda_\epsilon} = \frac{1}{2} + \frac{\epsilon}{12} + \ldots$

$d=3$ $\quad$ MF $\nu = .5$, $\nu_{sim} = 0.64$, $\nu_{expt} \cdot 6 - \cdot 7$ $\quad \nu_\epsilon \approx 0.58$