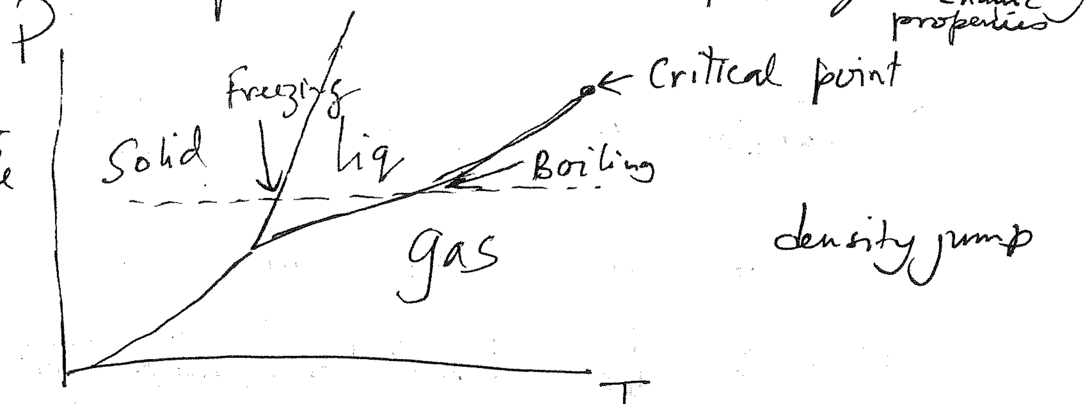
Phase Transitions and Critical Phenomena

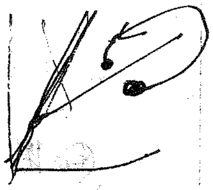
We have so far described non-interacting systems or interacting systems in one dimension. Interacting systems in higher dimension has interesting features like phase transition: abrupt changes in Thermodynamic properties

Familiar example



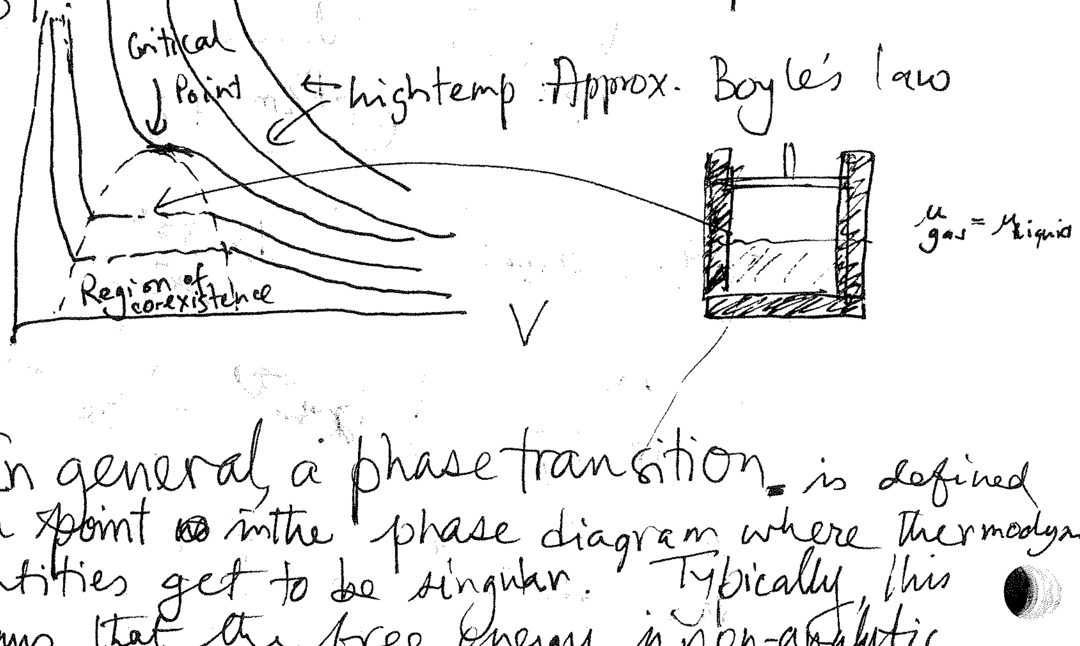
density jump

The solid phase ~~has~~ breaks translation invariance, liq & gas does not



Note that there is nothing qualitatively different between liquid and gas. The phase change is well defined but not the phases. In fact one could go from one phase to the other without seeing anything abrupt

Liquid Gas transition in PV plane

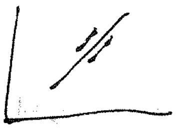


$\mu_{gas} = \mu_{liquid}$

In general, a phase transition is defined as a point in the phase diagram where thermodynamic quantities get to be singular. Typically, this means that the free energy is non-analytic there.

Classificn of phase transition: nth derivs of free energy singular ⟷ nth order transitn

In fact



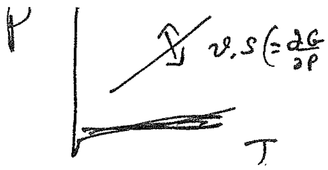
$d\mu_1 = d\mu_2$

$\Rightarrow -S_1 dT + v_1 dP = -S_2 dT + V_2 dP$

$\Rightarrow \frac{dP}{dT} = \frac{S_2 - S_1}{V_2 - V_1}$

$\frac{dP}{dT} = \frac{T(S_2-S_1)}{T(V_2-V_1)} = \frac{Q_v}{T(V_2-V_1)}$

Clausius Clapeyron eqn



$v, s \left(= \frac{\partial G}{\partial p}, \frac{\partial B}{\partial T}\right)$ discont. $\Rightarrow$ first order transition

Remember that $F = -k_BT \ln Z = -k_BT \ln\left(\sum_n e^{-\frac{E_n}{k_BT}}\right)$

Formally this quantity looks like an analytic function of T, except at T=0. How could we ever get phase transitions from a statistical mechanical description

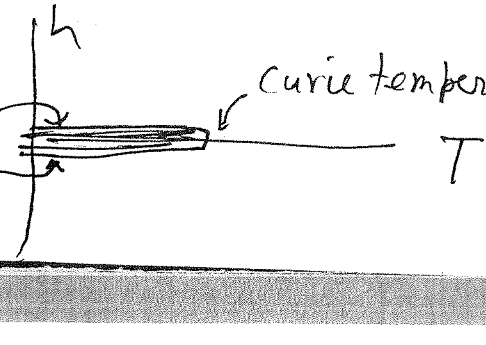
The formal statement is true provided the $\sum_n$ is finite (or controlled in some fashion).

Finite size systems (size L) with finitely many degrees of freedom have partition functions $Z_L(\beta)$ that are analytic in β (or T). However $\lim_{L\to\infty} \frac{\ln Z_L}{V_L}$ could still be singular.

Thus studying the statistical mechanics of phase transitions is a nontrivial endeavor. In only a few cases, we could prove rigorously that a phase transition happens at a particular temperature (and pressure, etc etc). More often we need indirect means and approximate methods to study phase transitions. One such method is the mean field theory (MFT)

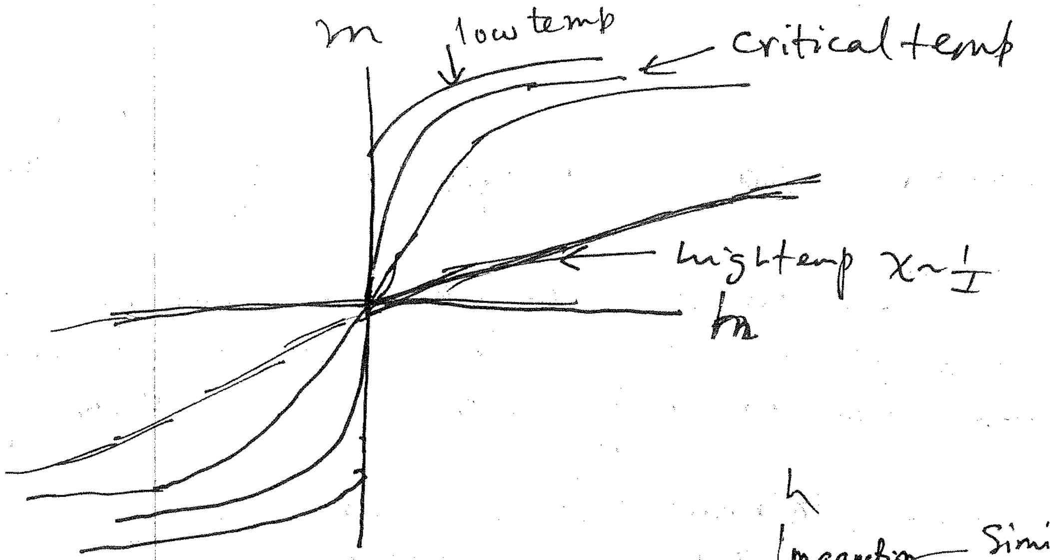
The simplest example to study MFT is the magnetic transition.

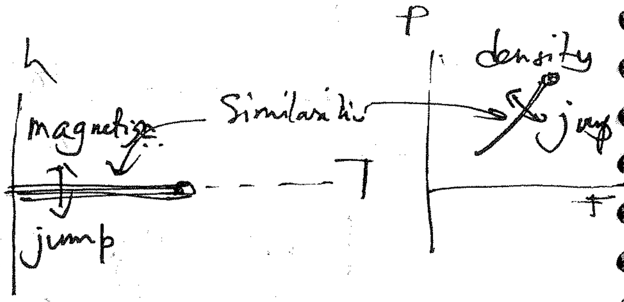
Ferromagnetic transition

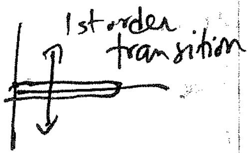


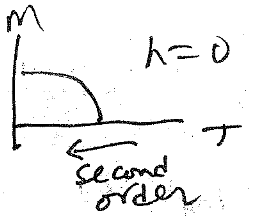
Curie temperature

nonzero magnetiz$^{\underset{..}{n}}$ at $h \to 0$









Ising model

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

Intuitive MFT: A single spin sees an effective field $h_{eff} = h + J \sum_{j \in NN} \langle \sigma_j \rangle$

If $\langle \sigma_j \rangle = m$ $\quad h_{eff} = h + zJm$

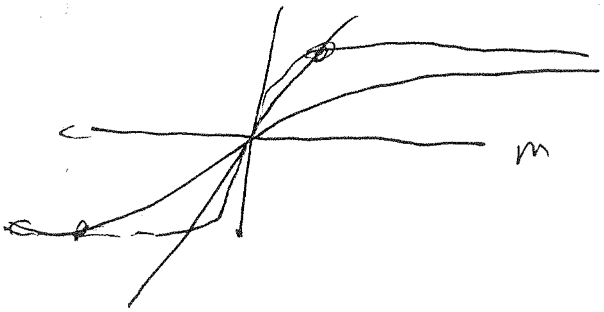
z being the # of neighbours in the lattice.

Self consistency $\quad m = th\, \beta h_{eff}$

$$= th\, \beta(zJm + h)$$

Solving this we could get m for a given h.

Note that for $h=0$ $m = \text{th}\,\beta z J m$ always has one solution, $m=0$



However for $\beta z J > 1$ there are two other solutions with $m \neq 0$. Spontaneous magnetization. These describe the ferromagnetic phase.

Let's call $T_c = zJ/k_B$ and $\beta_c = \frac{1}{zJ}$

When T is slightly above T_c, Let us calculate χ.

$$m = \text{th}(\beta(zJm+h)) \simeq \beta zJ m + \beta h = \frac{\beta}{\beta_c} m + \beta h$$

$$\Rightarrow \quad m = \left(1-\frac{\beta}{\beta_c}\right)^{-1}\beta h \simeq \frac{\beta_c^2}{\beta_c-\beta}\, h \simeq \frac{1}{k_B(T-T_c)}\, h$$

Thus $\frac{1}{k_B T}$ (curie law) gets modified to curie-Weiss law

$\chi \sim \frac{1}{T-T_c}$, showing a simple power law divergence of χ near the critical point.

Similarly for $h=0$ and T slightly less than T_c

$$m = \text{th}\,\beta zJ m = \beta zJ m - \frac{1}{3}(\beta zJ m)^3$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{\beta}{\beta_c}\right)^3 m^3 \simeq \left(\frac{\beta}{\beta_c}-1\right) m$$

$$\Rightarrow \quad m^2 = 3\frac{\beta-\beta_c}{\beta_c} \Rightarrow m \simeq \sqrt{3\frac{(T_c-T)}{T}}$$

So spontaneous magnetization $m \sim \sqrt{T_c-T}$. Another power law!

What about the MF estimate of free energy?

A more fancy way of getting the same thing is to go via the ~~[illegible]~~ variational principle [which as a by product produces an upper bound on Free energy.]

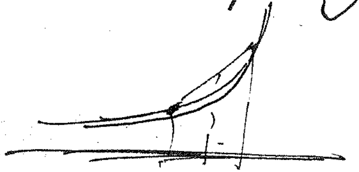
$$F \le \langle E \rangle_0 - T S_0$$

Where ~~Q~~ the subscript 0 refers to ~~to~~ a different Hamiltonian $E^{(0)}_n$; Every state $n \to E_n$

$\searrow E_n^{(0)}$

It is a application of Jensen inequality

$$\langle e^{x} \rangle \ge e^{\langle x \rangle}$$



Classical version

$$Z = \sum_n e^{-\beta E_n} = \sum_n e^{-\beta(E_n - E_n^{(0)})} e^{-\beta E_n^{(0)}}$$

$$= \left\langle e^{-\beta(E_n - E_n^{(0)})} \right\rangle_0 \times Z_0$$

$$Z_0 = \sum_n e^{-\beta E_n^0}$$

$$\left\langle e^{-\beta(E_n - E_n^0)} \right\rangle \ge e^{-\beta(\langle E \rangle_0 - E_0)}$$

$$F = -k_B T \ln Z \le \langle E \rangle_0 - E_0 + F_0$$

easy to calculate $\longrightarrow$ $= \langle E \rangle_0 - T S_0 \leftarrow$ easy to calculate

If there are a free parameter in $E^{(0)}_n$, the quantity $\langle E \rangle_0 - T S_0$ could be minimized to come closest to F.

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

$$E^{(0)} = - h_{eff} \sum_i \sigma_i$$

$$\langle E \rangle_0 - T S_0 = \langle E - E^{(0)} \rangle_0 + F_0$$

$$\langle E - E^{(0)} \rangle_0 = -\tfrac{1}{2} J N z m^2 - (h - h_{eff}) m$$

With

$$\langle \sigma_i \rangle = m = th(\beta h_{eff})$$

$$F_0 = -\frac{N}{\beta}\left[\ln\left(2 ch(\beta h_{eff})\right)\right]$$

$$F \leq N\left[-\tfrac{1}{2} J z m^2 - h m + h_{eff}\, m - \tfrac{1}{\beta} \ln 2 ch \beta h_{eff}\right]$$

$$= N \Phi(m)$$

Note that $m = \dfrac{e^{\beta h_{eff}} - e^{-\beta h_{eff}}}{e^{\beta h_{eff}} + e^{-\beta h_{eff}}} \Rightarrow \dfrac{1+m}{1-m} = e^{2\beta h_{eff}}$

$$\Rightarrow \frac{1}{2\beta} \ln\left(\frac{1+m}{1-m}\right) = h_{eff}$$

Aslo $2 ch \beta h_{eff} = e^{\beta h_{eff}} + e^{-\beta h_{eff}}$

$$= \sqrt{\frac{1+m}{1-m}} + \sqrt{\frac{1-m}{1+m}} = \frac{2}{\sqrt{1-m^2}}$$

$$\Phi(m) = -\tfrac{1}{2} J z m^2 - h m + \frac{m}{2\beta} \ln \frac{1+m}{1-m} - \frac{1}{\beta} \ln \frac{2}{\sqrt{1-m^2}}$$

~~How~~ What does the m (or h_{eff}) optimizn give

$$\Phi(m) = -\frac{1}{2}Jzm^2 - hm + \frac{1}{\beta}\left\{\frac{1+m}{2}\ln(1+m) + \frac{1-m}{2}\ln(1-m) - \ln 2\right\}$$

$$0 = \phi'(m) = -Jzm - h + \frac{1}{2\beta}\ln\frac{1+m}{1-m}$$

$$\Rightarrow \quad m = \text{th}\,\beta(Jzm+h) \qquad \text{same eqn.}$$

We could compute ~~However~~ the free energy by solving for m and putting it in $\Phi(m)$.

Note that mean field soln are minima of $\Phi(m)$. Let us look at T close to T_c and h small, making m small

$$\frac{1}{\beta}\left\{\frac{1+m}{2}\left(m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \cdots\right) - \frac{1-m}{2}\left(m + \frac{m^2}{2} + \frac{m^3}{3} + \frac{m^4}{4} + \cdots\right)\right\}$$

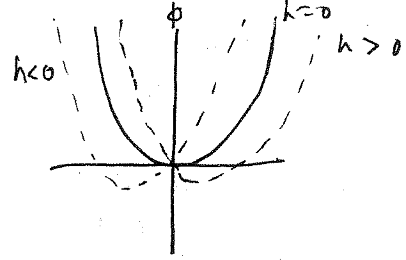
$$= \frac{1}{\beta}\left[\frac{m^2}{2} - \frac{m^4}{4} + m^2 + \frac{m^4}{3} \ldots\right]$$

$$= \frac{1}{\beta}\left(\frac{m^2}{2} + \frac{m^4}{12}\right)$$

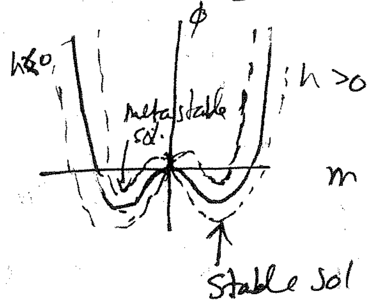
$$\Phi(m) = cst + \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\beta_c}\right) m^2 + \frac{1}{\beta}\frac{m^4}{12} - hm + O(m^6)$$

$$\simeq cst + \frac{k_B}{2}(T - T_c) m^2 + k_B T_c \frac{m^4}{12} - hm$$

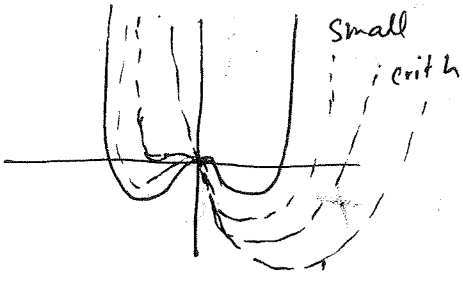
$T > T_c$



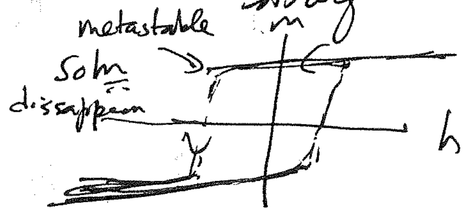
$T < T_c$



Large h for $T < T_0$



The metastable soln dissappears at some ~~point~~ field strength



Many of the properties around the critical point indept of explicit form of the original $\Phi(m)$.

Landau Theory: Order parameter [m for the mag problem]. Generic form of free energy consistent with symmetry [in our case $m \leftrightarrow -m$, when $h=0$]

The fact that within MFT, $\chi \sim \frac{1}{T-T_c}$ and spont mag goes as $\sqrt{T_c-T}$ could be derived from the very generic form of Landau free energy $\Phi = \frac{1}{2}k(\nabla m)^2 - hm + \frac{a}{2}(T-T_c)m^2 + \frac{b}{4}m^4 + \ldots$

Similarly, the free energy $+(T-T_c)m^2 + m^4$ optimizes to $m^2 \sim (T_c-T_c)$ when $T<T_c$ and 0 when $T>T_c$

$F \sim -(T-T_c)^2$ for $T<T_c$
0 " $T>T_c$
$\Rightarrow$ [illegible] $(T-T_c)$
C_V has a jump
$\alpha = 0$

We could also ask how correlation length depends upoun $(T-T_c)$. In order to do this within Landau theory, we need to write down the free energy of the system with ~~variable~~ spatially inhomogeneous m! The standard approach is the add a $\frac{1}{2}\kappa(\nabla m)^2$ term to the free energy. One could then calculate how a local field would affect the magnetiz'n profile around it.

Optimize $\frac{1}{2}\kappa(\nabla m)^2 + k_B(T-T_c)m^2 - h_0\delta(\vec{x})m + o(m^4)$

$$\Rightarrow \kappa\nabla^2 m + k_B(T-T_c)m = h_0\delta(\vec{x}) + o(m^3)$$

The soln $m \sim \frac{e^{-|x|/\xi}}{|x|}$ $\quad \xi^2 \sim \kappa/k_B(T-T_c)\}$

$$\xi \sim \frac{1}{\sqrt{T-T_c}}$$

Exptly, indeed, one finds powerlaws near the critical point: $\chi \sim |T-T_c|^{-\gamma}$, $C_V \sim (T-T_c)^{-\alpha}$, $m \sim (T_c-T)^{\beta}$, $\xi \sim (T-T_c)^{-\nu}$. But $\gamma \neq 1$, $\beta \neq \frac{1}{2}$, $\nu \neq \frac{1}{2}$, $\alpha \neq 0$ in general.

Troubles with MFT: Predicts phase transition in any dim (incl. 1d)

"Critical exponents" are dim indept.

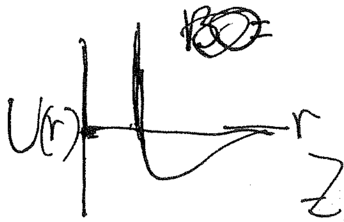
Keeping all these caveats in mind, the MFT is still a powerful weapon in statmech. As can be seen MFT is not single approach, but the choice of the Hamiltonian $H_0(E_0)$ affects what MFT we get.

Correction to ideal gas:

$$\mathcal{Z} = \sum_{N=0}^{\infty} z^N Z_N$$

$$\frac{PV}{k_BT} = \ln \mathcal{Z} = zZ_1 + z^2\left(Z_2 - \frac{1}{2}Z_1^2\right) + \cdots$$

$$Z_1 = \frac{V}{\lambda_T^3} \qquad z = e^{\mu/k_BT}$$



$$Z_2 = \frac{1}{2\lambda_T^6}\int d^3r_1\, d^3r_2\; e^{-U(r_1-r_2)/k_BT}$$

$$Z_2 - \frac{1}{2}Z_1^2 = \frac{1}{2\lambda_T^3}\int d^3\vec{r}_1\, d^3\vec{r}_2 \left(e^{-U(r_1-r_2)/k_BT} - 1\right)$$

$$= \frac{V}{2\lambda_T^3}\int d^3\vec{r}\left(e^{-U(r_1-r_2)/k_BT} - 1\right)$$

$$\int d^3r\left(e^{-U/k_BT} - 1\right) = -1 \times v_0 - \frac{1}{k_BT}\int_a^{\infty} 4\pi r^2 dr\, U(r) = -v_0 + \frac{y}{k_BT}$$

$$\frac{PV}{k_BT} = e^{\mu/k_BT}\,\frac{V}{\lambda_T^3} + e^{\frac{2\mu}{k_BT}}\,\frac{V}{\lambda_T^6}\left[-v_0 + \frac{y}{k_BT}\right]$$

$$N = e^{\frac{\mu}{k_BT}}\,\frac{V}{\lambda_T^3} + 2e^{\frac{2\mu}{k_BT}}\,\frac{V}{\lambda_T^6}\left[-v_0 + \frac{y}{k_BT}\right]$$

$$\frac{PV}{k_BT} = N - e^{\frac{2\mu}{k_BT}}\,\frac{V}{\lambda_T^6}\left[-v_0 + \frac{y}{k_BT}\right]$$

$$\simeq N - \frac{N^2}{V}\left[-v_0 + \frac{y}{k_BT}\right]$$

$$P = \frac{Nk_BT}{V}\left[1 + \frac{Nv_0}{V} - \frac{Ny}{Vk_BT}\right]$$

$$= \frac{Nk_BT}{V^2}\left[V + Nv_0\right] - \frac{N^2y}{V^2}$$

$$P + y\left(\frac{N}{V}\right)^2 \simeq \frac{Nk_BT}{V - Nv_0}$$

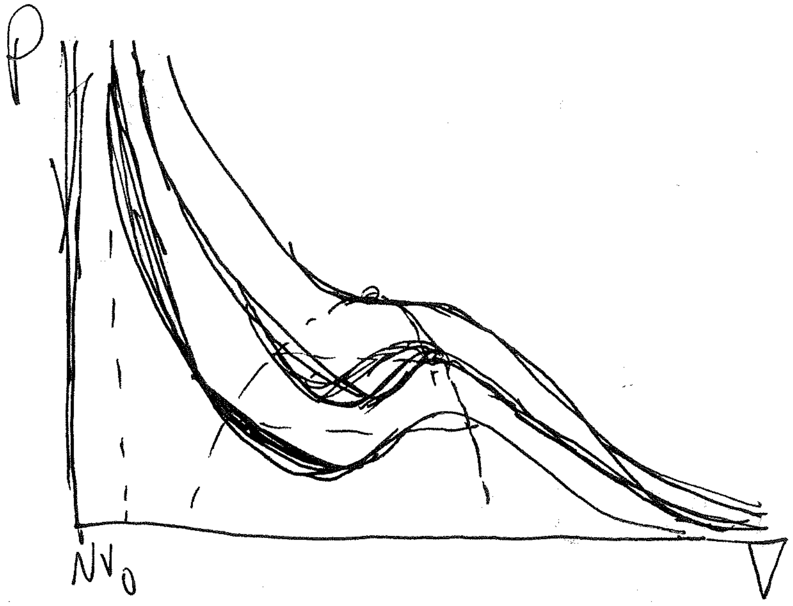
$$\left(P + y\underset{\substack{\uparrow\\ \text{effect}\\ \text{of the attractions}}}{\left(\frac{N}{V}\right)^2}\right)\left(V - \underset{\substack{\uparrow\\ \text{excluded}\\ \text{volume}\\ \text{effect.}}}{Nv_0}\right) = nRT$$

In general

$$\frac{PV}{Nk_B T} = 1 + \frac{B(T)N}{V} + B_2(T)\left(\frac{N}{V}\right)^2 + \cdots$$

Virial expansion!

B_1, B_2 virial coefficients



$$P = \frac{RT}{V-Nv_0} - \frac{yN^2}{V^2}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{(V-Nv_0)^2} + \frac{2yN^2}{V^3}$$

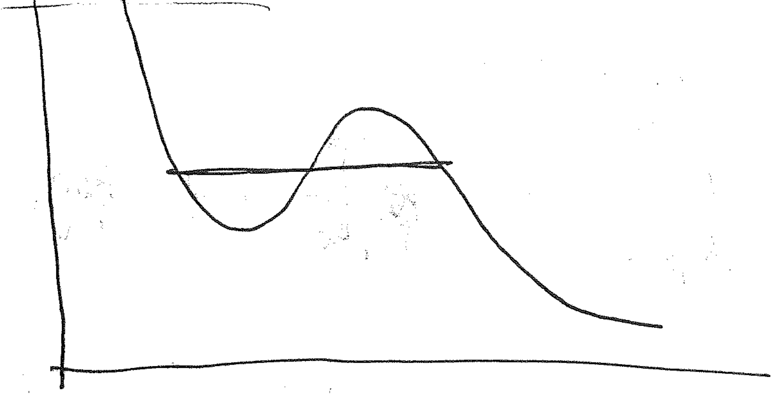
$$\frac{\partial P}{\partial V} = 0 \Rightarrow \frac{RT}{(V-Nv_0)^2} = \frac{2yN^2}{V^3}$$

Note that two roots merge if in addition $\frac{2RT}{(V-Nv_0)^3} = \frac{6yN^2}{V^4} \Rightarrow \frac{V-Nv_0}{2} = \frac{V}{3}$

$V_c = 3Nv_0$, $\frac{RT_c}{(2Nv_0)^2} = \frac{2yN^2}{(3Nv_0)^3} \Rightarrow RT_c = \frac{8yN}{27v_0} \Rightarrow P_c = \frac{4y}{27v_0^2} - \frac{yN^2}{9N^2v_0^2} = \frac{y}{27v_0^2}$

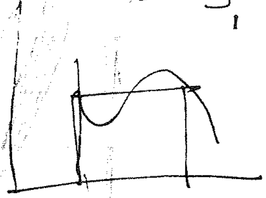
$$\left(P + \frac{3P_c V_c^2}{V^2}\right)\left(V - \frac{1}{3}V_c\right) = \frac{8 P V_c}{3 T_c} T$$

$$\left(\left(\frac{P}{P_c}\right) + \frac{3}{(V/V_c)^2}\right)\left(3\left(\frac{V}{V_c}\right) - 1\right) = 8\left(\frac{T}{T_c}\right)$$



$$\int_1^2 d\mu = 0$$

$$\Rightarrow \int_1^2 V dP = 0 \Rightarrow P(V_2 - V_1) = \int_1^2 P dV$$



~~T_c~~

$$\left(P' + 3/V'^2\right)\left(3V' - 1\right) = 8T'$$

$$\left[1 + \delta P' + \frac{3}{(1+\delta V')^2}\right]\left[2 + 3\delta V'\right] = 8 + 8\delta T'$$

$$\left[4 + \delta P' + 3(1 - 2\delta V' + 3\delta V'^2 - 4\delta V'^3 + \dots)\right]\left[2 + 3\delta V'\right] = 8 + 8\delta T'$$

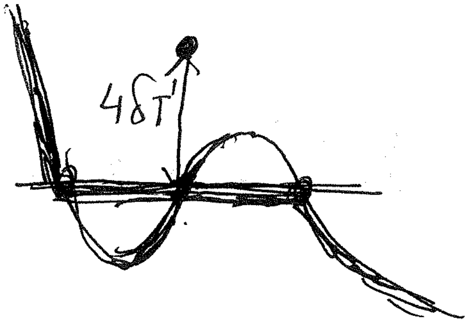
$$\left(4 + \delta P' - 6\delta V' + 9\delta V'^2 - 12\delta V'^3 + \dots\right)\left(2 + 3\delta V'\right) = 8 + 8\delta T'$$

$$2\delta P' - \cancel{12\delta V'} + 18\delta V'^2 - 24\delta V'^3 + \cancel{12\delta V'} + 3\delta P'\delta V' - \cancel{18\delta V'^2} + 27\delta V'^3 + \dots = 8\delta T'$$

$$\delta P'(2+3\delta v') + 3\delta v'^3 = 8\delta T'$$

$$\delta P' = \frac{1}{2}\left(8\delta T' - 3\delta v'^3\right)\left(1 - \frac{3}{2}\delta v' + \dots\right)$$

$$= 4\delta T' - 6\delta T'\delta v' - \frac{3}{2}\delta v'^3 + \dots$$



$$\delta P' - 4\delta T' = -6\delta T'\delta v' - \frac{3}{2}\delta v'^3$$

Note that for small $\delta P', \delta v, \delta T$ setting $\delta P' = 4\delta T'$ takes care of Maxwell construction.

$$-6\delta T'\delta v'_{1,2} - \frac{3}{2}\delta v'^3_{1,2} = 0$$

$$\delta v'_{1,2} = \pm\sqrt{\frac{2}{3}\times 6\times(-\delta T')} = \pm 2\sqrt{|\delta T|}$$

$$\Delta v = 2\times 2\sqrt{|\delta T'|} = 4\sqrt{|\delta T'|}$$

[Very similar to $\Delta m_{spont} \sim |\delta T|^{\frac{1}{2}}$]

Similarly, above the transition

$$\frac{\partial P'}{\partial v'} = -6\delta T' - 3\delta v'^2 + \dots \quad \text{Compressibility} - \frac{1}{v'}\frac{\partial v'}{\partial P'} \sim \frac{1}{6\delta T'}$$

[Similar to $\chi \sim \frac{1}{T-T_c}$]

Note that the magnetic system and the liq. gas system may not be so different! Take Ising model

$$E = -J\sum_{\langle ij\rangle} \sigma_i \sigma_j - h\sum \sigma_i$$

If if take up spin to be an occupied site and down spin to be an empty site

$$\sigma_i = 2n_i - 1 \qquad n_i = 0, 1$$

$$E_{Ising} = -4J\sum_{\langle ij\rangle} n_i n_j + (4J - h)\sum_i n_i$$

If we call $4J = U$ and $4J - h = -\mu$

$$E_{gas} - \mu N = E_{Ising} = -\mu\sum n_i - U\sum_{\langle ij\rangle} n_i n_j$$

Then Ising spin distribution corresponds to the grand canonical ensemble of a a lattice gas. $n = 0, 1$ ⇒ hard core repulsion.

Ferromagnetic J ⇒ nearest neighbor attraction.

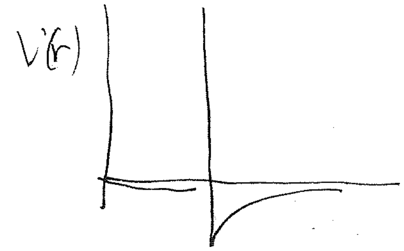
Ferromagnetism ⇒ Phase separation
liquid-gas transition.

van der Waals gas

$$Z = \int \frac{dp^{3N}}{h^N} e^{-p^2/2mk_BT} Z_{conf} \qquad Z_{conf} = \int dr^{3N} e^{-U(r_1 \cdots r_N)}$$

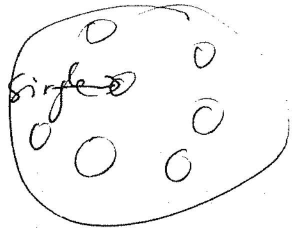
The kinetic contr $\int \frac{d^{3N}p}{h^N} e^{-p^2/2mk_BT} = \frac{1}{\lambda_T^{3N}}$

Int. pot hard core + shallow attraction



$$F_{conf} \leq \langle E_{conf} \rangle_0 - TS_0$$

Consider the distrn to be all confs' allowed by hard sphere constraints



$$S_0 \approx k_B \ln (V - Nv_0)^N$$

$$\langle E_{conf} \rangle_0 = \frac{1}{2} \int_{r > a_0} N \times \left(\frac{N}{V - v_0}\right) U(r) d^3r$$

$$\simeq \frac{1}{2} \frac{N^2}{V} \int_{r > a_0} U(r)\, d^3r$$

$$F \leq -\frac{\gamma N^2}{V} - Nk_B T \ln (V - Nv_0)$$

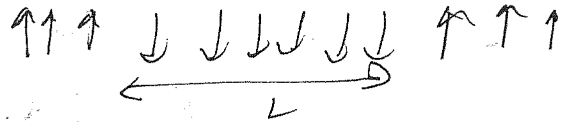
$$P = -\frac{\partial F}{\partial V} = -\frac{\gamma N^2}{V^2} + \frac{Nk_BT}{V - Nv_0}$$

[After the discussion of cont. symmetry systems]

Phase Transitions beyond MFT.

Low temp expn : Start with the ordered state

1d



Making an inverted domain costs finite amnt of energy.

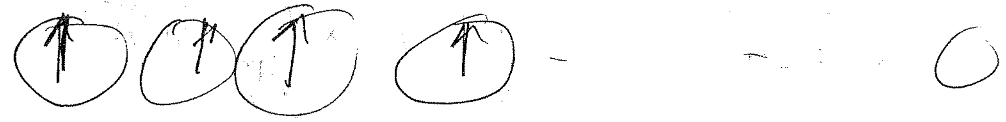
L^2 domain to flip at finite cost

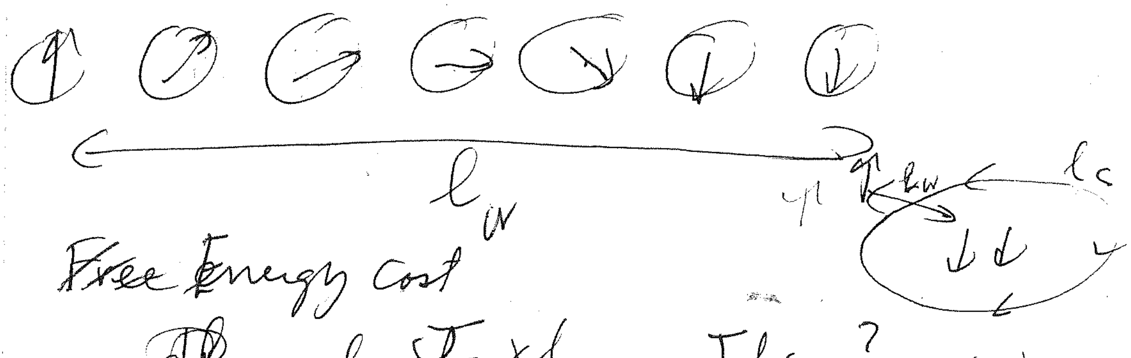
$\sum e^{-\beta \Delta E}$

At any finite β this would happen.

2d ~~Disor~~ Depends upon symmetry

$E = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$





~~Free~~ Energy cost

$\cancel{dl}$ $l_w \frac{J}{l_w^2} \times l_s \sim J \frac{l_s}{l_w} \overset{?}{\sim} O(1)$

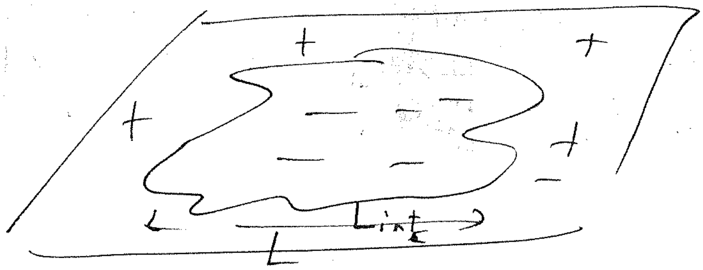
or perhaps going as log L

Ⓠ So slowly varying changes causing disruption of ordered state with very little energy cost.

Goldstone ~~boson~~ modes destroy order state in two dimension.

This is the point of the Mermin Wagner theorem.

For discrete systems



The energy cost $\sim L_{int}(L)$

The entropy $\sim L$ aswell.
(log of # possible boundaries)

If L_{int}/L stays constant or diverges then, we could argue that the ordered state is stable.

We will give this argnement carefully for Ising model [Peierls argument]

Move up by twopages

[Should have been done first, before general discussion.]

Heisenberg magnets $E = -J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j$

$$\vec{S}_i^2 = 1$$

$$H_0 = -\vec{h}_{eff} \cdot \vec{S}$$

$$|\vec{m}| = \frac{\int d\theta \sin\theta \cos\theta \, e^{\beta h_{eff} \cos\theta}}{\int d\theta \sin\theta \, e^{\beta h_{eff} \cos\theta}} = \int_{-1}^{1} dx \, x \, e^{\beta h_{ex} x} \Big/ \int_{-1}^{1} dx \, e^{\beta h_{ex} x}$$

Now $\int_{-1}^{1} dx \, e^{\gamma x} = \frac{e^{\gamma} - e^{-\gamma}}{\gamma} = Z(\gamma)$

$$\frac{\int dx \, x \, e^{\gamma x}}{\int dx \, e^{\gamma x}} = \frac{\partial \ln Z(\gamma)}{\partial \gamma} = \coth\gamma - \frac{1}{\gamma}$$

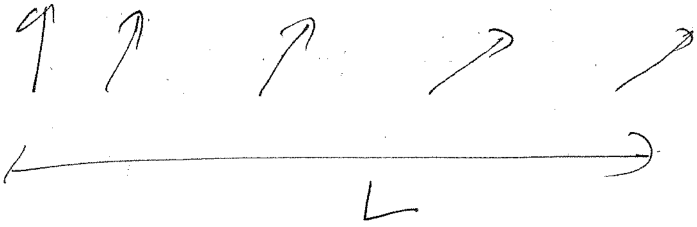
$$\coth\gamma = \frac{1}{\gamma} + \frac{\gamma}{3} + O(\gamma^3)$$

Criterion for critical point

$$\Rightarrow \quad 1 = \frac{1}{3}\beta_c J z$$

$$\boxed{T_c = \frac{zJ}{3k_B}}$$

In systems with cont symm, the ordered state that breaks the symmetry has low lying excitations called goldstone modes



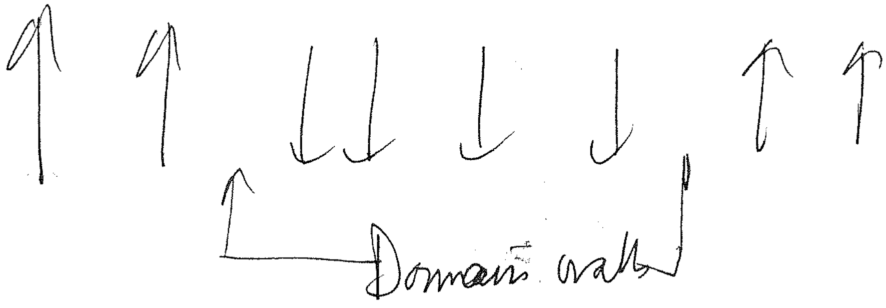
$$S_1 \cdot S_2 \sim \cos\theta \sim \cos\frac{\alpha}{L} \sim 1-\frac{\alpha^{2}}{L^2}$$

$$E \sim V \times \frac{\alpha^2}{L^2}$$

~~If $\alpha^2 \sim \frac{1}{V}$ the energy contr. $\sim \frac{1}{L^2}$~~

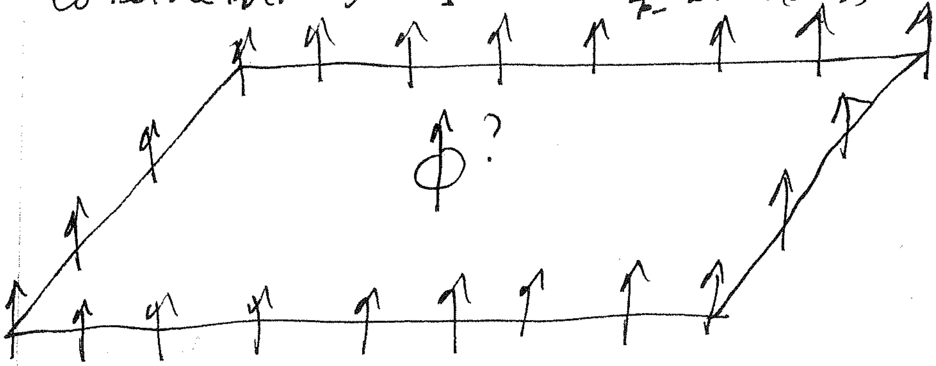
$$e^{-\beta V \times \frac{\alpha^2}{L^2}} \Rightarrow \alpha^2 \sim \frac{k_B T \times L^2}{V}$$

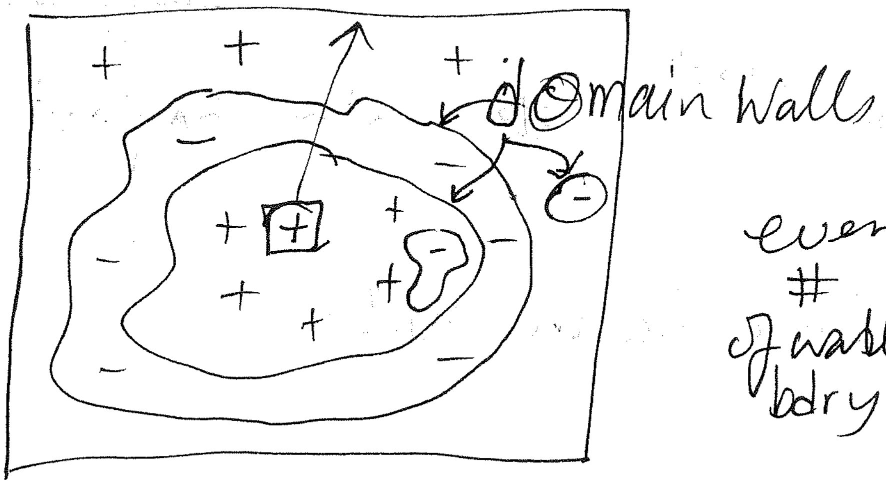
In a discrete system



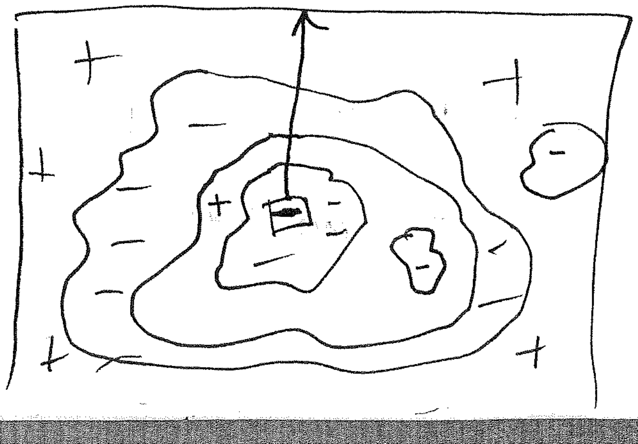
Peierl's argument for an ordered state in the twodimensional Ising model.

If the system is ferromagnetic, small biases like the bdry being up spin, would affect the prob of being upspin deep inside. In the paramag phase spins correlate over length ξ and $\frac{p_+}{p_-} \approx 1 + o(e^{-L/\xi})$





even # of walls to bdry



odd # to boundary.

cost of domain walls $e^{-2K \times \text{Length}}$ $K = \beta J$

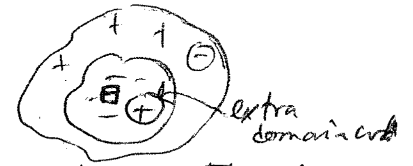
$+ | -$ $\quad$ $+ \; +$

$+J$ $\quad$ $-J$

$\Delta E = 2J$

$e^{-2\beta J}$

$$\frac{p_-}{p_+} \le \frac{\sum_{\text{outer}} \sum_{\text{inner}}' e^{-2K L_{\text{inner}}} \, e^{-2K L'_{\text{outer}}}}{\sum_{\text{outer}} e^{-2K L'_{\text{outer}}}}$$



Inequality, when the − domain is not simply connected.

We want to get an upper bound on p_- / p_+ that could be <1 as $L \to \infty$

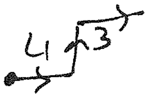
~~Find~~ # of inner wall of length L

$\le$ # " " " " " " without considering constraints of outer walls

$= g(L)$

$$\frac{p_-}{p_+} \le \frac{\sum_{\text{outer}} \sum_{L} g(L) e^{-2KL} \, e^{-2KL'}}{\sum_{\text{outer}} e^{-2KL'}} = \sum_L g(L) e^{-2KL}$$

Lets count $g(L)$



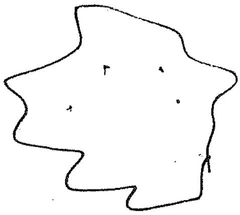
Start from a point

$4\times3^{L-1}$ walks.

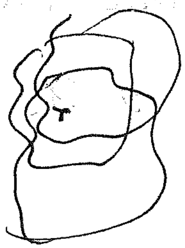
Not all close. so # closed walks without backtracking

with a point marked and an orientation of walk. $\le 4\times3^{L-1}$

For loops the starting from any where is fine and any orientation is fine. So $\frac{4\times3^{L-1}}{2L}$

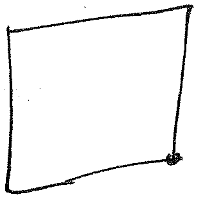


For a particular shape Loop, how many would enclose the spin? # Points in it.



$$g(L)\le \#\text{of points inside the loop}\ \frac{4\times3^{L-1}}{2L}$$

$$\le \left(\frac{L}{4}\right)^2\times\frac{4\times3^{L-1}}{2L}$$



Biggest area loop

Far worse

So $$\frac{p_-}{p_+} \leq \sum_L g(L)\, e^{-2KL} \leq \sum_L \frac{L}{8}\, 3^{L-1} e^{-2KL}$$

This sum should run ~~of~~ over only even L but lets even relax that

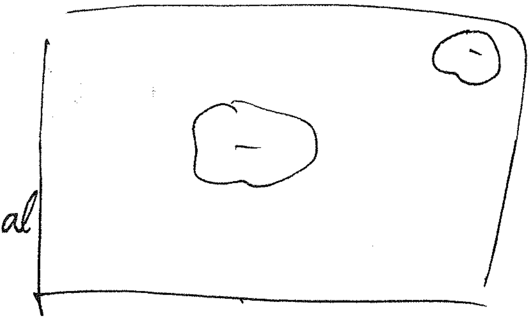
$$\frac{e^{-2K}}{8}\left[1 + 2\left(3e^{-2K}\right) + 3\left(3e^{-2K}\right)^2 + \cdots\right]$$

$$= \frac{e^{-2K}}{8\left(1-3e^{-2K}\right)^2} \to 0 \quad \text{as } K \to \infty$$

~~As e^{-2K} becomes small~~ For K=1, sum

$$\frac{e^{-2}}{8\left(1-3e^{-2}\right)^2} = \frac{0.135}{8\times\left(1-3\times0.135\right)} \simeq 0.04$$

At low temp domain walls are rare. Their combinatorial possibilities suppressed by $e^{-2K\times\text{Length}}$.



$\Rightarrow$ Low temp expn:

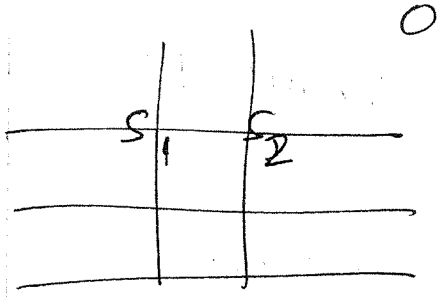
$$\sum_{\text{walls}} e^{-2K\times(\text{Length of walls})}$$

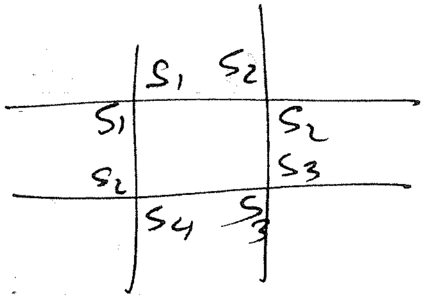
In fact, the other way of calculating things is from high T lim

$$\sum_{\{S\}}\prod_{\langle ij\rangle} e^{K S_i S_i} = \sum_{\{S\}}\prod_{\langle ij\rangle}(\text{ch}K + S_iS_j\,\text{sh}K)$$

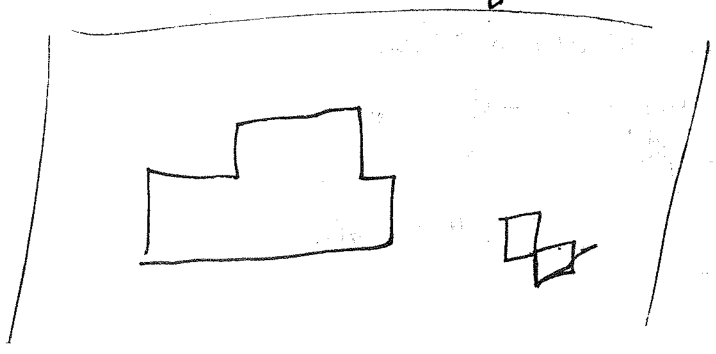
$$= 2^{L^2}(\text{ch}K)^{\frac{L3}{2}} \sum_{\{S\}}\prod_{ij}(1+S_iS_j\,\text{th}K) \Big/ \sum_{\{S\}} 1$$

$$1 + \underbrace{\sum_{\langle ij\rangle}\langle \text{th}\cancel{K}S_iS_j\rangle_0}_{0} + (\text{th}K)^N \sum_{\substack{\langle ij\rangle\\ \langle kl\rangle}}\langle S_iS_jS_kS_l\rangle_0$$





$$(\text{th}K)^4 \sum\langle \underbrace{S_1^2S_2^2S_3^2S_4^2}_{1}\rangle$$

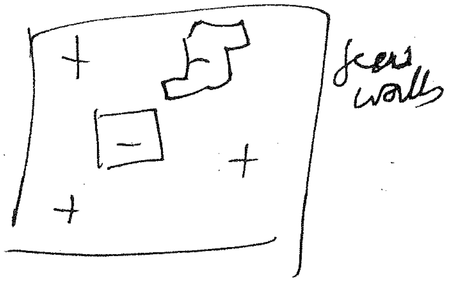


loops of "correlated spin"

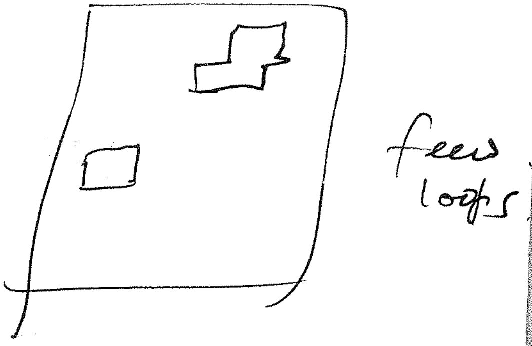
$$Z = 2^{L^2}(\mathrm{ch}K)^{\frac{L3}{2}} \sum_{\text{loops}} (\mathrm{th}K)^{\text{Length loops}}$$

At high temp K~~=βJ~~ is small, so thK is small
So we have very few loops.

Low temp



~~Th~~ High temp



$$Z(K) \longleftrightarrow Z(\tilde{K})$$

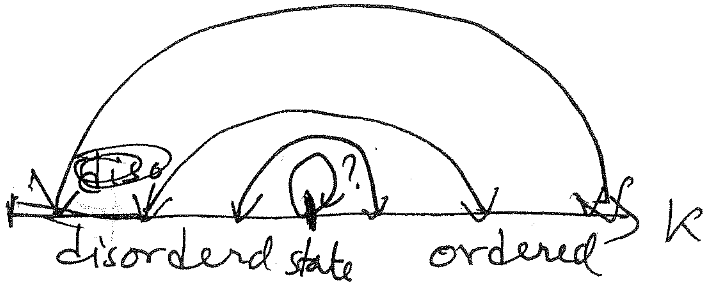
$$e^{-2K} = \mathrm{th}\tilde{K} = \frac{\mathrm{sh}\tilde{K}}{\mathrm{ch}\tilde{K}}$$

Better way $2\mathrm{sh}2K = e^{2K} - e^{-2K} = \frac{\mathrm{ch}\tilde{K}}{\mathrm{sh}\tilde{K}} - \frac{\mathrm{sh}\tilde{K}}{\mathrm{ch}\tilde{K}}$

$$= \frac{\mathrm{ch}^2\tilde{K} - \mathrm{sh}^2\tilde{K}}{\mathrm{sh}\tilde{K}\,\mathrm{ch}\tilde{K}^2}$$

$$= \frac{1}{\frac{1}{2}\mathrm{sh}2\tilde{K}}$$

$$\boxed{\mathrm{sh}2K\,\mathrm{sh}2\tilde{K} = 1}$$



Could the critical point be self dual?
[Unless there are multiple ~~top~~ transitions] Yes!

So 2d ~~tri~~ criticality of Ising

$$\sinh 2K \cancel{\times} \mathrm{sh}\, 2K = 1$$

or $\sinh 2\beta J = 1$

$$e^{2K} - e^{-2K} = 2 \Rightarrow e^{2K} + e^{-2K} = \sqrt{(e^{2K} - e^{-2K})^2 + 4}$$
$$= \sqrt{8} = 2\sqrt{2}$$

or $e^{2K} = 1 + \sqrt{2}$ $\quad K = \frac{1}{2}\ln(\sqrt{2}+1)$
$= 0.44\ldots$

$$k_B T_c = \frac{2J}{\ln(\sqrt{2}+1)} = 2.2692 \times J$$

The details of the mag. model does not matter! Even magnets and fluids are similar. UNIVERSALITY

Critical Phonomena, scaling, universality

Widom Scaling
Generalized homogenous fnc.

$$f_{sing}(\lambda^p \varepsilon, \lambda^q h) = \lambda f_{sing}(\varepsilon, h) \qquad \varepsilon = T - T_c$$

$$\underset{spnt}{m} \sim \frac{\partial f}{\partial h}\Big|_{h=0} \sim |\varepsilon|^\beta \qquad m \sim \frac{\partial f}{\partial h}\Big|_{\varepsilon=0} \sim |h|^{1/\delta}.$$

Set $\lambda = \varepsilon^{-\frac{1}{p}}$

$$\Rightarrow f_{sing}(\varepsilon, h) = \varepsilon^{\frac{1}{p}} f_{sing}\left(1, \frac{h}{\varepsilon^{q/p}}\right)$$

$$\Rightarrow m \sim \frac{\partial f}{\partial h} \sim \varepsilon^{\frac{1-q}{p}} f_2\left(1, \frac{h}{\varepsilon^{q/p}}\right)$$

When $h \to 0$ $\quad m \sim \varepsilon^{\frac{1-q}{p}}$ $\quad \boxed{\beta = \frac{1-q}{p}}$

When $\varepsilon \to 0$ $f(1,x)$ has to go as $x^{\frac{1}{\delta}}$, ~~so that~~ but ε dependence cancels if $\frac{1-q}{p} = \frac{q}{p} \times \frac{1}{\delta}$

$$\boxed{\delta = \frac{q}{1-q}}$$

MF $\beta = \frac{1}{2}$ $\delta = 3$ $\quad$ Expts m 3d mag $\beta \simeq 0.3{-}0.4$ $\quad \delta \simeq 4{-}5$