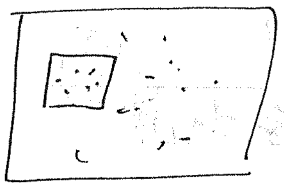
~~thlbt~~ The grand canonical ensemble



Number of particles not conserved anymore

$$\Omega(E_{system}, N_{system}, E_{bath}, N_{bath})$$

$$E_{system} + E_{bath} = E_T$$
$$N_{system} + N_{bath} = N_T$$

$$(\ )_{system} \ll (\ )_T$$

$$\Omega_{system}(E, N)\ \underbrace{\Omega_{bath}(E_T - E, N_T - N)}$$

$$e^{\ln \Omega_{bath}(E_T - E, N_T - N)}$$

$$\exp\left[\ln \Omega_{bath}(E_T, N_T) - E\frac{\partial \ln\Omega}{\partial E} - N\frac{\partial \ln\Omega}{\partial N} + \cdots\right]$$

$$\Omega_{system}(E, N)\ \Omega(E_T, N_T)\ e^{-\beta E + \beta\mu N}$$

✓

$$\text{Prob} = \text{const}\ \Omega_{system}(E, N)\, e^{-\beta(E - \mu N)}$$

$$T\,dS = dU + P\,dV - \mu\,dN$$

$$T\frac{\partial S}{\partial N} = -\mu$$

$$\text{or}\quad \frac{\partial \ln\Omega}{\partial N} = -\frac{\mu}{k_B T} = -\beta\mu$$

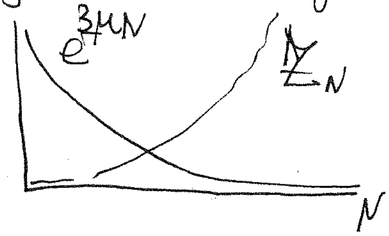
Grand canonical partition fn

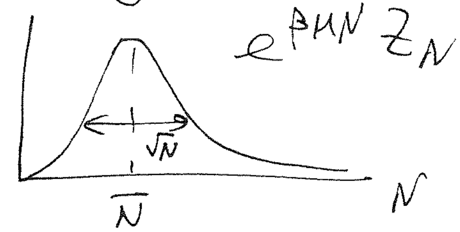
$$\mathcal{Z} = \sum_{N, E_n^{(N)}} e^{\beta\mu N} e^{-\beta E_n^{(N)}}$$

$$= \sum_N e^{\beta\mu N} Z_N = \sum_N z^N Z_N$$

$z = e^{\beta\mu}$ is called the fugacity

For negative $\beta\mu$, say, we have the following familiar situ-





μ determined by fixing the condn that $\bar{N}$ is the av # of particles.

Note that under that circumstance,

$$\mathcal{Z} \simeq e^{\beta\mu\bar{N}} Z_{\bar{N}} \times (\text{order } \sqrt{N})$$

$$\ln \mathcal{Z} \simeq \underbrace{\beta\mu\bar{N} - \beta F_{\bar{N}}}_{\beta[G-F]} + O(\ln N)$$

$$\beta PV$$

$$\boxed{\ln \mathcal{Z} = \frac{PV}{k_B T}}$$

~~Math~~ Multispecies version

$$\mathcal{Z} = \sum_{N_1,\dots,N_p} e^{\beta\sum_{i=1}^{p}\mu_i N_i} Z_{N_1,\dots,N_p}$$

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The same reln holds for $\ln \mathcal{Z}$, i.e., $\boxed{\frac{PV}{k_BT} = \ln \mathcal{Z}}$

Let us now look at special cases.

Ideal Gas (Classical)

$$\sum_N e^{\beta\mu N} Z_N$$

$$= \sum_N e^{\beta\mu N} \frac{1}{N!} \left(\frac{V}{\lambda_T^3}\right)^N$$

$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$

$$\left(\frac{e^{\beta\mu}V}{\lambda_T^3}\right)^N \frac{1}{N!}$$ peaks at

$$N \simeq \frac{e^{\beta\mu}V}{\lambda_T^3},$$ since after that $(\,)^N$ grows by the same fact by $\frac{1}{N!}$ falls down faster b..

$$\Rightarrow \boxed{\mu = k_BT \ln \frac{\bar{N}\lambda_T^3}{V} = k_BT \ln \rho\lambda_T^3}$$

[We already got that before]

Note that $\mathcal{Z} = \exp\left[e^{\beta\mu}V/\lambda_T^3\right] = \exp \bar{N}$

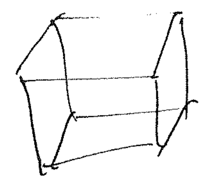
$\bar{N} = \ln \mathcal{Z} = \frac{PV}{k_BT}$

Quantum version: density matrix

$$\hat{\rho} = \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}} = \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{\mathcal{Z}}$$

Quantum gases

Particles in a box



States specified by momenta

$$\vec{k} = \left(\frac{2\pi m_x}{L_x}, \frac{2\pi m_y}{L_y}, \frac{2\pi m_z}{L_z}\right) \Rightarrow \text{energy } \epsilon_{\vec{k}}$$

$\epsilon_k = c|\vec{k}|$ for massless particles

$\epsilon_k = \frac{\hbar^2 \vec{k}^2}{2m}$ for non-rel. massive particles

$n_{\vec{k}}$ is the number of particles at momentum $\vec{k}$.

Bosons $\quad n_{\vec{k}} = 0, 1, 2, 3, \ldots$

Fermions $\quad n_{\vec{k}} = 0, 1$

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$= \sum_{\{n\}} e^{-\beta\left(\sum_{\vec{k}} \epsilon_{\vec{k}} n_{\vec{k}} - \mu \sum_{\vec{k}} n_{\vec{k}}\right)}$$

$$= \prod_{\vec{k}} \sum_{n_{\vec{k}}} e^{-\beta(\epsilon_{\vec{k}}-\mu) n_{\vec{k}}}$$

$$\mathcal{Z}_{Bose} = \prod_{\vec{k}} \left[1 + e^{-\beta(\epsilon_k - \mu)} + e^{-2\beta(\epsilon_k-\mu)} + \cdots \right] \quad ?$$

$$= \prod_{k} \frac{1}{1 - e^{-\beta(\epsilon_k - \mu)}}$$

Note that $\langle n_k \rangle = \dfrac{0 + e^{-\beta(\epsilon_k-\mu)} + 2e^{-2\beta(\epsilon_k-\mu)} + \cdots}{1 + e^{-\beta(\epsilon_k-\mu)} + e^{-2\beta(\epsilon_k-\mu)}}$

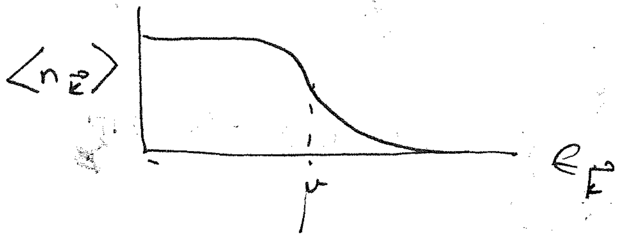
$$= \frac{1}{\beta}\partial_\mu \ln\left(1 + e^{-\beta(\epsilon_k-\mu)} + e^{-2\beta(\epsilon_k-\mu)} + \cdots\right)$$

$$= \frac{1}{\beta}\partial_\mu \ln \frac{1}{1 - e^{-\beta(\epsilon_k-\mu)}}$$

$$= \frac{1}{\beta}\frac{\beta \times e^{-\beta(\epsilon_k-\mu)}}{1 - e^{-\beta(\epsilon_k-\mu)}} = \frac{1}{e^{\beta(\epsilon_k-\mu)} - 1}$$

$$\mathcal{Z}_F = \prod_k \left[1 + e^{-\beta(\epsilon_k - \mu)} \right]$$

$$\langle n_k \rangle = \frac{e^{-\beta(\epsilon_k-\mu)}}{1 + e^{-\beta(\epsilon_k-\mu)}} = \frac{1}{e^{\beta(\epsilon_k-\mu)} + 1}$$



Connection to ideal gas: When z is small ($\beta\mu$ large negative)

$$\frac{PV}{k_BT} = \ln \mathcal{Z} = \sum_k \ln(1 + z e^{-\beta\epsilon_k}) = z\sum_k e^{-\beta\epsilon_k} - \frac{1}{2}z^2 \sum_k e^{-2\beta\epsilon_k} + \cdots \Big\} \text{Fermion}$$

$$= -\sum_k \ln(1 - z e^{-\beta\epsilon_k}) = z\sum_k e^{-\beta\epsilon_k} + \frac{1}{2}z^2\sum_k e^{-2\beta\epsilon_k} + \cdots \Big\} \text{Boson}$$

$$N = \sum_k \frac{z e^{-\beta\epsilon_k}}{1+z e^{-\beta\epsilon_k}} = z\sum_k e^{-\beta\epsilon_k} - z^2\sum_k e^{-2\beta\epsilon_k} + \cdots \quad \text{\} Fermion}$$

$$= \sum_k \frac{z e^{-\beta\epsilon_k}}{1-z e^{-\beta\epsilon_k}} = z\sum_k e^{-\beta\epsilon_k} + z^2\sum_k e^{-2\beta\epsilon_k} + \cdots \quad \text{\} Boson}$$

$$\frac{PV}{Nk_BT} = 1 \pm \frac{z\sum_k e^{-2\beta\epsilon_k}}{\sum_k e^{-\beta\epsilon_k}} \quad \begin{matrix} \text{Fermion} \\ \text{Boson} \end{matrix}$$

$$P_{Fermi} > P_{classical} > P_{Bose}$$

Applications of Bose Einstein Statistics

Black body radiation.

$\omega_k = c|k|$ c is the speed of light

$$\sum_{\vec{k}} = \sum_{\{m\}} \approx \int dm_x\, dm_y\, dm_z = \frac{L_x L_y L_z}{(2\pi)^3}\int dk_x\, dk_y\, dk_z$$

$$= V\int \frac{d^3k}{(2\pi)^3} = V\int \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{V\omega^2 d\omega}{2\pi^2 c^3}$$

A factor of 2 for that two polarization
No particle number constraints for photons $\mu = 0$

$$\frac{V\omega^2 d\omega}{\pi^2 c^3}\,\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} = n(\omega)\,d\omega$$

The energy density $\hbar\omega\, n(\omega)\, d\omega$

$$\frac{V\hbar\,\omega^3 d\omega}{\pi^2 c^3}\,\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}$$ is Planck's law

Often written as

Energy density $$\frac{8\pi\nu^2 d\nu}{c^3}\,\frac{h\nu}{e^{\frac{h\nu}{k_BT}}-1}$$

Total energy density

$$\int_0^\infty \frac{8\pi\nu^2 d\nu}{c^3}\,\frac{h\nu}{e^{\frac{h\nu}{k_BT}}-1}$$

$$= \frac{8\pi(k_BT)^4}{c^3h^3}\underbrace{\int_0^\infty \frac{x^3\,dx}{e^x-1}}$$

$$\int dx\,x^3\left(e^{-x}+e^{-2x}+e^{-3x}+\dots\right) = 3!\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right] = 3!\,\zeta(4)$$

$$= \frac{\pi^4}{90}\times 6$$

So $\rho = \dfrac{6\times\cancel{8}4\pi^5 k_B^4}{45h^3c^3}\,T^4 = \dfrac{\pi^2 k_B^4}{15c^3\hbar^3}\,T^4$

$$= \underbrace{\sigma}_{\text{Stefan Boltzmann const}}\,T^4$$

BTW, remember that for Debye phonons $U \sim T^4$. Same math!

Note that if $\hbar \to 0$ $\sigma \to \infty$

$$\frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \approx k_B T \qquad \text{(equipartition)}$$

$$\rho \sim \int_0^\infty \frac{8\pi \nu^2 d\nu}{c^3} k_B T$$: Divergence from large ν

QM cuts off the integral at $\nu \sim \frac{k_B T}{h}$

$$\frac{8\pi}{c^3} \int_0^{k_B T/h} \nu^2 d\nu \; k_B T \sim (\;) \frac{k_B^4 T^4}{c^3 h^3}$$

This is why Plack introduced h in the first place. Bose noted that he could think of photons as identical particles and derive Planck's law. Einstein extended Bose's argument to nonrelativistic particles.

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}$$

μ chosen so that $\quad \mu \le 0 \quad \mu \to 0$ increases the number

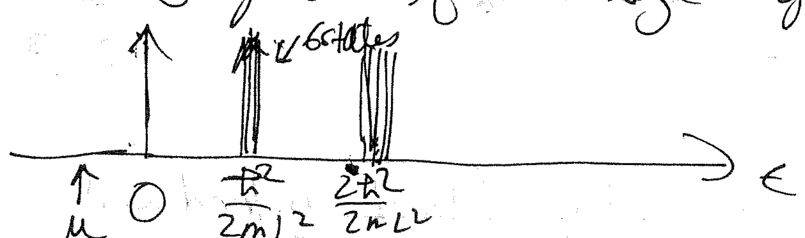
$$N = \frac{V}{2\pi^2} \int \frac{k^2 dk}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} - 1}$$

He noted that the integral tends to a finite number when $\mu \to 0$

$\frac{1}{2\pi} \int_0^\infty \frac{k^2 dk}{e^{\beta \hbar^2 k^2 / 2m} - 1}$ is finite setting a density $\frac{N}{V}$ above which something interesting happens.

What happens is known as Bose Einstein Condensation. The only way to satisfy the constraints for N/V above the threshold is to have a macroscopic number of particles at the $\vec{k}=0$ state.

Why does the $\vec{k}=0$ state get chosen? Think of the spectrum finite size system with $L_x=L_y=L_z=L$



If $\beta\mu \sim -\frac{1}{(\Delta\rho)L^3}$ the occupancy of the state $\vec{k}=0$ is $\frac{1}{e^{-\beta\mu}-1} \simeq -\frac{1}{\beta\mu} = \Delta\rho L^3$

The occupance of the next sets of states is

$$n = \frac{1}{\beta\left(\frac{\hbar^2}{2mL^2} + \frac{1}{\Delta\rho L^3}\right)}$$

For large L, $\frac{1}{\Delta\rho L^3} << \frac{\hbar^2}{2mL^2}$ and so

$$n_k \simeq \frac{1}{e^{\beta\epsilon_k}-1}$$

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Note that for small $\vec{k}$ $\quad n_k \sim \frac{1}{k^2}$
but $\int k^2 dk\, n_k$ is not divergent at the small k end.

$$N = \frac{z}{1-z} + \frac{4\pi V}{h^3}\int_{\hbar/L}^{\infty} p^2 dp \, \frac{z}{e^{\beta p^2/2m} - z}$$

$z = e^{\beta\mu}$

~~Note that~~ Going variable $x = \sqrt{\frac{\beta}{2m}}\, p$

$$N = \frac{z}{1-z} + \frac{4\pi V}{h^3}(2mk_BT)^{3/2}\int_0^\infty \frac{z x^2 dx}{e^{x^2}-z}$$

$$= \frac{z}{1-z} + \frac{4V}{\sqrt{\pi}\,\lambda_T^3}\int x^2 dx \left[\frac{z}{e^{x^2}-z}\right]$$

$$\int_0^\infty x^2 dx \, \frac{z}{e^{x^2}-z} = \int_0^\infty dx \left[z x^2 e^{-x^2} + z^2 x^2 e^{-2x^2} + z^3 x^2 e^{-3x^2} + \dots\right]$$

$$= \sum_{n=1}^\infty \frac{z^n}{n^{3/2}} \times \underbrace{\int_0^\infty dy\, y^2 e^{-y^2}}$$

$y^2 = w$

$\frac{1}{2}\int_0^\infty dw\, w^{1/2} e^{-w}$

$= \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{4}\sqrt{\pi}$

$$g_{3/2}(z) = \sum_{n=1}^\infty \frac{z^n}{n^{3/2}}$$

So $N = \frac{z}{1-z} + \frac{V}{\lambda_T^3} g_{3/2}(z)$

Note that at the transition $z \to 1$

$$N = \frac{V}{\lambda_T^3} g_{3/2}(1) = \frac{V}{\lambda_T^3} \zeta(3/2)$$

$$\frac{N}{V}\lambda_T^3 = \rho \lambda_T^3 = \zeta(3/2) = 2.612\ldots$$

So we need roughly 2.6 particles in λ_T^3 volume

Let us now figure out λ_T for an atom like Na-23 [11 protons + 11 electrons + 12 neutrons = 44 fermions $\equiv$ a boson]

$$\frac{h}{\sqrt{2\pi m k_B T}} = \sqrt{\frac{h^2}{2\pi m k_B T}}$$

$$= \sqrt{\frac{(6.6 \times 10^{-34}\ \mathrm{J\,s})^2}{2 \times 3.14 \times 23 \times 1.66 \times 10^{-27}\,\mathrm{kg} \times 1.38 \times 10^{-23}\,\mathrm{J K^{-1}} \times T}}$$

$$\approx \left(\frac{40 \times 10^{-68}\ \mathrm{J^2 s^2}}{150 \times 2 \times 10^{-50}\ T\,\mathrm{kg\,J K^{-1}}}\right)^{1/2}$$

$$\sim \left(\frac{4 \times 10^{-19}\ \mathrm{m^2 K}}{3\ T}\right)^{1/2} \sim \left(\frac{13\,K}{T}\right)^{1/2} \mathring{A}$$

Trouble is that when the separation between atoms ~ few Å we get strong interaction.

~~He-4 / ... Ma-23 ...~~

~~the weight~~ He-4 does become superfluid at about 2K, but needs interaction for description.

Need much larger λ_T to avoid interaction

Recent techniques (1995, Ketterle at MIT, Cornell and Weiman at JILA) used dilute gases $\frac{N}{V} \sim 10^{14}\,cm^{-3}$

We need $\lambda_T \sim \sqrt[3]{10^{-14}}$ cm

$\sim 2\times10^{-5}$ cm $\sim 2000\,\mathring{A}$

$T < \frac{13\,K}{4\times10^{6}} \sim 3\,\mu K$ [This where condensate starts forming]

[JILA group used Rb-87

$\lambda_T \sim \left(\frac{3\,K}{T}\right)^{1/2} \mathring{A}$

$T \sim 170\,nK$ $\qquad \lambda_T \simeq \left(\frac{1}{6\times10^{-8}}\right)^{1/2} \mathring{A} \sim \frac{10^4}{2}\,\mathring{A}$

$\sim 4000\,\mathring{A}$]

For explicit calculations of various properties, like internal energy:

$$\ln \mathcal{Z}_G = -\ln(1-z) - \frac{4\pi V}{h^3}\int p^2 dp \ln\left(1-z e^{-p^2/2mk_BT}\right) = -\ln(1-z) + \frac{V}{\lambda_T^3}\, g_{5/2}(z)$$

$$U = \frac{4\pi V}{h^3}\int_{h/L}^{\infty} p^2 dp \frac{z}{e^{\beta p^2/2m} - z} \times \frac{p^2}{2m} = \frac{4V}{\sqrt{\pi}\lambda_T^3} \times \frac{1}{\beta}\int_0^\infty \frac{z x^4 dx}{e^{x^2} - z} = \frac{4V}{\sqrt{\pi}\lambda_T^3} \times k_BT \times \int dy\, y^4 e^{-y^2} \times g_{5/2}(z)$$

$$= \frac{3}{2} k_BT \times \frac{V}{\lambda_T^3} \times g_{5/2}(z)$$

Application of Fermi distribution

Electrons in a metal

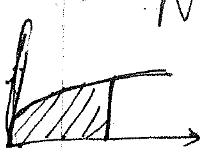
Let us do it for an arbitrary density of states $\rho(\epsilon)$. For free electrons of mass m

$$\rho(\epsilon) = V\int \frac{d^3k}{(2\pi)^3}\,\delta\left(\epsilon - \frac{\hbar^2k^2}{2m}\right)$$

$$= \frac{V}{2\pi^2}\int_0^\infty k^2 dk\,\delta\left(\epsilon - \frac{\hbar^2k^2}{2m}\right) = \frac{V}{2\pi^2}\int \frac{m}{\hbar^2 k}\,\delta\left(k - \sqrt{\frac{2m\epsilon}{\hbar^2}}\right) k^2 dk$$

$$= \frac{V}{2\pi^2}\,\frac{m}{\hbar^2}\sqrt{\frac{2m\epsilon}{\hbar^2}} = \frac{Vm}{2\pi^2\hbar^3}\sqrt{2m\epsilon}$$

$$N = \int \frac{\rho(\epsilon)\,d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \qquad U = \int \frac{\rho(\epsilon)\,\epsilon\,d\epsilon}{e^{\beta(\epsilon-\mu)}+1}$$

At $T=0$ 

$\mu_0 = \epsilon_F$

If we change T, μ

$$dN = \frac{dT}{k_BT^2}\int \frac{e^{\beta(\epsilon-\mu)}(\epsilon-\mu)\rho(\epsilon)\,d\epsilon}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2} + \frac{d\mu}{k_BT}\int \frac{e^{\beta(\epsilon-\mu)}\rho(\epsilon)\,d\epsilon}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2}$$

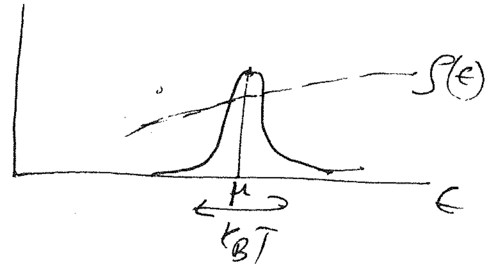
$$= \frac{dT}{k_BT^2}\int \frac{d\epsilon\,\rho(\epsilon)(\epsilon-\mu)}{\left[2\,\mathrm{ch}\,\frac{\beta(\epsilon-\mu)}{2}\right]^2} + \frac{d\mu}{k_BT}\int \frac{d\epsilon\,\rho(\epsilon)}{\left[2\,\mathrm{ch}\,\frac{\beta(\epsilon-\mu)}{2}\right]^2}$$

$$dU = \frac{dT}{k_BT^2}\int \frac{d\epsilon\,\rho(\epsilon)(\epsilon-\mu)\,\epsilon}{[\quad]^2} + \frac{d\mu}{k_BT}\int \frac{d\epsilon\,\rho(\epsilon)\,\epsilon}{[\quad]^2}$$

$$= \frac{dT}{k_BT^2}\int \frac{d\epsilon\,\rho(\epsilon)(\epsilon-\mu)^2}{[\quad]^2} + \frac{d\mu}{k_BT}\int \frac{d\epsilon\,\rho(\epsilon)(\epsilon-\mu)}{[\quad]^2}$$

$$+ \mu\,dN$$

Note that $\left(\cosh \frac{\beta(\epsilon-\mu)}{2}\right)^{-2}$



$$dU = \frac{dT}{k_BT^2}\left\{\rho(\mu_0)\,(k_BT)^3\int_{-\infty}^{\infty}\frac{t^2dt}{(2Ch\,t/2)^2} + \rho''(\mu_0)\times O((k_BT)^5)\right\}$$

$$+ \frac{d\mu}{k_BT}\left\{\rho'(\mu_0)\cancel{(k_BT)}\, O((k_BT)^3)\right\} + \mu\, dN$$

$dN = 0$

$$C_V = \frac{\partial U}{\partial T} = k_B^2 T\, \rho(\mu_0)\times\frac{\pi^2}{3} = \frac{\pi^2 k_B^2}{3}\rho(\mu_0)\, T$$

Since
$$\int_{-\infty}^{\infty}\frac{t^2dt}{(2ch\,t/2)^2} = 2\int_0^{\infty}\frac{e^t\,t^2dt}{(1+e^t)^2} = 2\left[\int e^{-t}t^2dt - 2\int e^{-2t}t^2dt + 3\int e^{-3t}t^2dt + \cdots\right]$$

$$= 2\times 2!\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$= 4\left[\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\} - \frac{2}{2^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\}\right]$$

$$= 4\times\left(1-\frac{1}{2}\right)\times\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$= 2\,\zeta(2) = 2\times\frac{\pi^2}{6} = \frac{\pi^2}{3}$$

$$\int_{-\infty}^{\infty}\frac{dt}{(2ch\,t/2)^2} = 2\left[\int_0^{\infty}dt\,e^{-t} - 2\int_0^{\infty}dt\,e^{-2t} + 3\int\cdots + \cdots\right] = \text{trouble, But} \int_{-\infty}^{\infty}\frac{e^x dx}{(1+e^x)^2} = \int_0^{\infty}\frac{dz}{(1+z)^2} = -\frac{1}{1+z}\Big|_0^{\infty} = 1$$

$z = e^x$

For free particle

$$C_V = \frac{\pi^2 k_B^2}{3} T \times \frac{Vm}{2\pi^2\hbar^3}\sqrt{2m\epsilon_F}$$

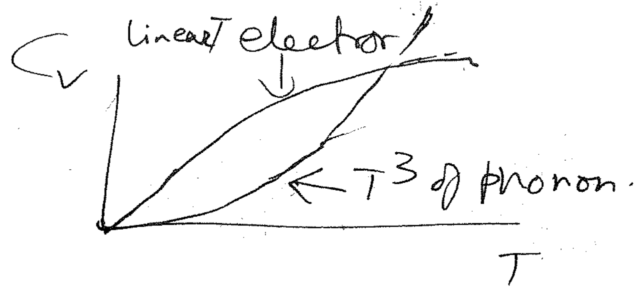
$$= \frac{V k_B^2 m \sqrt{2m\epsilon_F}}{6\hbar^3} T$$

At zero T

$$N = \frac{Vm}{2\pi^2\hbar^3}\sqrt{2m}\times\frac{2}{3}\epsilon_F^{3/2}$$

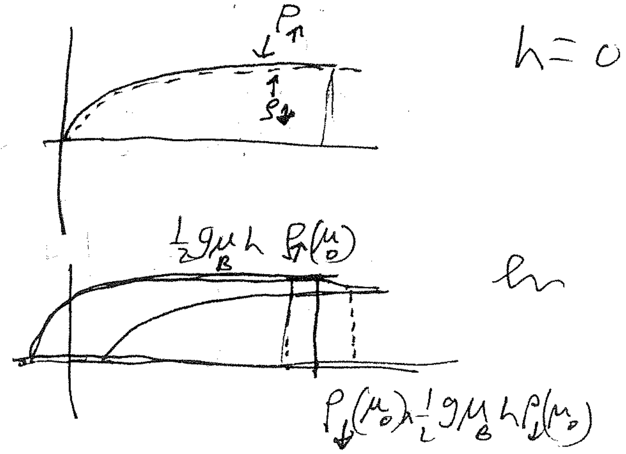
$$C_V = \frac{N\pi^2}{2}\,\frac{k_B^2 T}{\epsilon_F} = \frac{\pi^2}{3}\left(\frac{k_B T}{\epsilon_F}\right)\times\underbrace{\frac{3}{2}N k_B}_{\text{ideal classical gas}}$$

Note

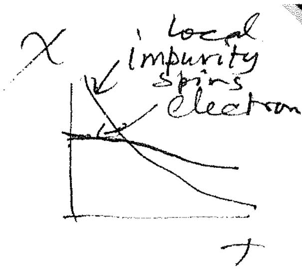


Now think of the spins: electrons have spin 1/2

$$E = \frac{\hbar^2 k^2}{2m} + g\mu_B h \frac{\sigma^z}{2}$$



$$\Delta m_{\uparrow\downarrow} = \left(\pm\rho(\mu)\,\frac{1}{2}g\mu_B h\right)\times\left(\pm\frac{g}{2}\mu_B\right) \Rightarrow m = \frac{1}{2}g^2\mu_B^2\rho(\mu)\,h$$



$$\chi = \frac{1}{2} g^2 \mu_B^2 \rho(\mu) \,, \quad \text{Compare this to Curie} \quad \frac{1}{4} g^2 \mu_B^2 / k_B T$$

$$C_V = 2 \times \frac{\pi^2 k_B^2}{3} \rho(\mu)\, T \qquad \text{(for two spin states)}$$

~~Compare~~ Also note that $\quad \dfrac{T\chi}{C_V} = \dfrac{3}{4} \dfrac{g^2 \mu_B^2}{\pi^2 k_B^2}$

If you really wanted to calculate $\mu(T)$

$$dN = 0 \Rightarrow \frac{dT}{k_B T^2} \int \frac{d\epsilon\, \rho(\epsilon)(\epsilon-\mu)}{[2\cosh \beta\frac{(\epsilon-\mu)}{2}]^2} + \frac{d\mu}{k_B T} \int \frac{d\epsilon\, \rho(\epsilon)}{[2\cosh \beta\frac{(\epsilon-\mu)}{2}]^2} = \frac{dT}{k_B T^2} (k_B T)^3 \rho' \int \frac{dx\, x^2}{(2\cosh\frac{x}{2})^2} + \frac{d\mu}{k_B T}(k_B T) \int \frac{dx}{(2\cosh \frac{x}{2})^2}$$

In general

$$\boxed{\Rightarrow d\mu = -k_B \times \frac{\int \frac{dx\, x^2}{(2\cosh\frac{x}{2})^2}}{\int \frac{dx}{(2\cosh\frac{x}{2})^2}} \times \frac{\rho'}{\rho} \times T\, dT}$$

$$= -k_B \frac{\pi^2}{3} \frac{\rho'}{\rho} T\, dT$$

$$\mu = \mu_0 - \frac{\pi^2 \rho'}{6 \rho} k_B T^2$$

$$N = \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} + 1}$$

$$= \frac{V}{h^3} \int d^3p \frac{z}{e^{\beta p^2/2m} + z}$$

$$k = \frac{p}{\hbar} = \frac{2\pi p}{h}$$

$$= \frac{4\pi V}{h^3} (2 m k_B T)^{3/2} \int dx \frac{z x^2}{e^{x^2} + z}$$

$$= \frac{4V}{\sqrt{\pi}\, \lambda_T^3} \int dx\, x^2 \left[z e^{-x^2} - z^2 e^{-2x^2} + z^3 e^{-3x^2} - \ldots \right]$$

$$= \frac{4V}{\sqrt{\pi}\, \lambda_T^3} \underbrace{\int_0^\infty dy\, y^2 e^{-y^2}}_{\frac{1}{4}\sqrt{\pi}} \sum_n \frac{(-1)^n z^n}{n^{3/2}} = \frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n^{3/2}}$$

$$= \frac{V}{\lambda_T^3} f_{3/2}(z)$$

$$\boxed{\ln \mathcal{Z} = \frac{V}{\lambda_T^3} f_{5/2}(z)}$$