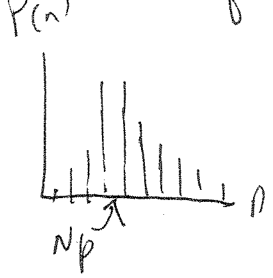
Some Probability theory results

Prob. Distr$^{\underline{n}}$: Discrete ~~&~~ and continuous

Binomial Distr$^{\underline{n}}$: N independent trials, each having two outcomes: "success" (1), or "failure" (0). The probability of getting n successes



$$P(n) = {}^{N}C_{n} \; p^{n} \; q^{N-n} \qquad \boxed{q = 1-}$$

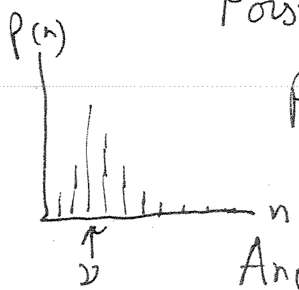
${}^{N}C_{n}$: # of ways n successes could be chosen from N trial

p^{n}: prob of n successes in a specific order

q^{N-n}: Prob of N−n fa

One can show that $\langle n \rangle = Np$, $Var(n) = \langle n^2 \rangle - \langle n \rangle^2 = N$

If there are many trials ($N \to \infty$) and success in each trial is very unlikely ($p \to 0$), in such a way that $Np = \nu$ stays fixed, then we get Poisson distribution out of Binomial distr$^{\underline{n}}$.

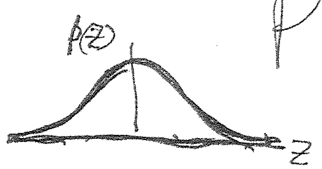


$$P(n) = e^{-\nu} \frac{\nu^{n}}{n!}$$

Another interesting limit of binomial distr$^{\underline{n}}$ ~~is~~ as N is defining the variable

$$Z = (n - Np)/\sqrt{Npq}$$

As $N \to \infty$, Prob distr n of Z gets well approximated by a normal distrn



$$P(z_1 \le Z \le z_2) = \int_{z_1}^{z_2} dx \; \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

For prob. distr$\underset{..}{\text{ns}}$ taking real values we could calculate something called the characteristic function

$$F(k) = \langle e^{ikx} \rangle = \int dx\, p(x)\, e^{ikx}$$

Fourier transform of the prob distr$\underset{..}{\text{n}}$ [or $\sum_n p(n)\, e^{ikn}$ for the discrete example]

Knowing $F(k)$ is eqvt. to knowing $p(x)$ since

$$p(x) = \int \frac{dk}{2\pi} F(k)\, e^{-ikx}$$

This is related to generating function for integer valued distr$\underset{..}{\text{ns}}$ $\sum p(n) z^n$ with z replaced by e^{ik}.

One use of characteristic function is calculation of moments

$$F(k) = \langle e^{ikx} \rangle = \left\langle \sum \frac{(ik)^n x^n}{n!} \right\rangle = \sum_n \frac{(ik)^n}{n!} \langle x^n \rangle$$

So the Taylor exp$\underset{..}{\text{n}}$ of $F(k)$, provided it exists, gives the moments of the distribution.

One last comment about characteristic functions: If we have two random variables X_1 & X_2, with char. function $F_1(k)$ and $F_2(k)$, then the char. function for the

Sum $X = X_1 + X_2$ is $F(k) = F_1(k)\, F_2(k)$.

The proof is just an application of convolution thm from Fourier transform.

$$\mathrm{Prob}(X = x) = \int dx_1 dx_2\, p_1(x_1)\, p_2(x_2)\, \delta(x - x_1 - x_2)$$
$$= \int dx_1\, p_1(x_1)\, p_2(x - x_1) = (p_1 * p_2)(x)$$

$$\langle e^{ikx} \rangle = \int dx\, e^{ikx} \int dx_1\, p_1(x_1)\, p_2(x - x_1)$$
$$= \int dx_1 \underbrace{\int dx\, e^{ik(x-x_1)}\, p_2(x - x_1)}\, e^{ikx_1}\, p_1(x)$$
$$= \int dx_1\, F_2(k)\, e^{ikx_1}\, p_1(x)$$
$$= F_1(k)\, F_2(k)$$

If $X = X_1 + X_2 + \cdots + X_N$,
$$F(k) = F_1(k)\, F_2(k) \cdots F_N(k)$$

This relation would be very important in studying the distribution of the sum of a large number of random varibles.
By the way, how much is $F(0)$?
$$F(0) = \langle 1 \rangle = 1$$

Now let us get some practice calculating $F(k)$ for various distributions

BTW, $\log F(k) = ik \times \text{mean} + \frac{(ik)^2}{2} \times \text{Variance} + O(k^3)$

Binomial: $F(k) = \sum_n {}^N C_n\, p^n q^{N-n} e^{ikn} = (pe^{ik} + q)^N$

$$F(k) = \left[p(1 + ik - \tfrac{k^2}{2} + O(k^3)) + q\right]^N = [\underbrace{p+q}_{1} + p(ik - \tfrac{k^2}{2}) + O(k^3)]^N$$

$$= 1 + Np(ik - \tfrac{k^2}{2}) + \frac{N(N-1)}{2} p^2 (ik - \tfrac{k^2}{2})^2 + O(k^3)$$

$$= 1 + iNpk + \left\{ -\frac{N(N-1)}{2} p^2 k^2 - Np\frac{k^2}{2} \right\} + O(k^3)$$

$$= 1 + ik\, Np + \frac{(ik)^2}{2} \{ N^2p^2 + Np(1-p) \} + O(k^3)$$

$\langle n \rangle = Np \qquad \langle n^2 \rangle = Np(1-p) + (Np)^2$

$\mathrm{Var}(n) = \langle n^2 \rangle - \langle n \rangle^2 = Np(1-p) = Npq$

Poisson: $F(k) = \sum_n e^{-\nu} \frac{\nu^n}{n!} e^{ikn} = e^{\nu(e^{ik}-1)}$

$\log F(k) = \nu(e^{ik} - 1) = \nu(ik + \frac{(ik)^2}{2} + \cdots)$

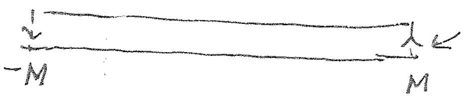
mean $= \nu$, variance $= \nu$, ...

Gaussian: $$F(k) = \int dx\, \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} e^{ikx} = e^{ik\mu} \int \frac{dy}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2} + iky}$$

$$= e^{ik\mu} e^{\frac{(ik\sigma^2)^2}{2\sigma^2}} \int e^{-(y + ik\sigma^2)^2/2\sigma^2} \frac{dy}{\sigma\sqrt{2\pi}}$$

$$= e^{ik\mu + \frac{(ik)^2}{2}\sigma^2}$$

We need $$\int_{-\infty}^{\infty} dy\, e^{-(y+ia)^2/2\sigma^2} = \int_{-\infty+ia}^{+\infty+ia} dz\, e^{-z^2/2\sigma^2} = \int_{-\infty}^{\infty} dy\, e^{-y^2/2\sigma^2}$$



$a e^{-M^2/2\sigma} \to 0$ as $M \to \infty$

Note that Poisson $\ln F(k) = \nu\left(ik - \frac{k^2}{2} - i\frac{k^3}{6} + \cdots\right)$

True of assymetric distrn in general

For symm. case: Binomial at $p = q = \frac{1}{2}$ $\quad F(k) = e^{ikN/2}\left(2\,\mathrm{ch}\frac{k}{2}\right)^N = \left(2e^{\frac{ikN}{2}}\right) e^{-\frac{N}{2}\left(\frac{k}{2}\right)^2 - \frac{N\left(\frac{k}{2}\right)^4}{12}} + \cdots$

Sum of i.i.d variables

[redacted] $Y = X_1 + \cdots + X_N$, each X_i is independent and identically distributed with pdf $p(x)$

Char. fnc. of each X_i is $f(k) = \int e^{ikx} p(x)\,dx$

Characteristic function of [redacted] Y

$$F(k) = \left(f(k)\right)^N$$

Let $f(k) = e^{ik\mu + \frac{(ik)^2}{2}\sigma^2 + O(k^3)}$

$$F(k) = e^{ikN\mu + \frac{(ik)^2}{2}N\sigma^2 + O(Nk^3, Nk^4)}$$ [redacted]

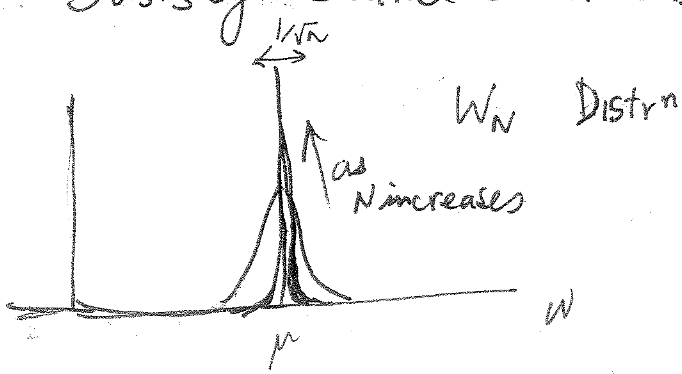
Mean $\sim N\mu$ $\quad$ variance $\sim N\sigma^2$ $\quad$ std. dev $\sim \sqrt{N}\,\sigma$

If we define $W_N = \frac{[redacted]\,Y}{N} = \frac{1}{N}(X_1 + \cdots + X_N)$

mean$(w) = \mu$ $\quad$ Std. dev $(w) = \frac{\sigma}{\sqrt{N}}$

↳ gets smaller as $N \to \infty$

Basis of Central Limit thms



Weak law of large numbers: $\quad$ Uncorrelated [redacted] variables with μ, σ^2

$$\lim_{N \to \infty} \mathrm{Prob}\left(|W_N - \mu| < \varepsilon\right) = 1$$

$$\boxed{\text{Stronger version: } \operatorname{Prob}\left(\lim_{N\to\infty} W_N = \mu\right) = 1 \text{ provided } X_i\text{'s are i.i.d with } \langle |X_i| \rangle \text{ are finite}}.$$

Once more consider i.i.d variables with well defined μ and σ^2. [redacted] Greater understanding comes from asking limiting form of the distribution of the departure of W_N from μ. Since we expect this to be of order $1/\sqrt{N}$, let us define

$$Z_N = \frac{W_N - \mu}{\sigma/\sqrt{N}}$$

One could show $\lim_{N\to\infty} \operatorname{Prob}(Z_N \le z) = \Phi(z)$

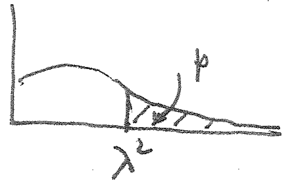
where $\Phi(z)$ is the error fnc. $\Phi(z) = \int_{-\infty}^{z} dy \, \frac{e^{-y^2/2}}{\sqrt{2\pi}}$

This is the central limit theorem.

Proof of the weak version of law of large numbers depend of Chebyshev inequality

$$\operatorname{Prob}(|X-\mu| > k\sigma) \le \frac{1}{k^2}$$

Proof
Consider the distr$\underline{n}$ of $|X-\mu|^2$



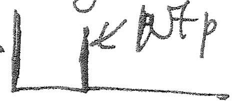
p

λ^2

Say $\operatorname{prob}(|x-\mu|^2 \ge \lambda^2) = p$ what is the smallest $\langle |x-\mu|^2 \rangle$, we could get

subject to that constraint

1-p wt at 0



wt p at λ^2

So $\sigma^2 \ge p\lambda^2$.

Choose $p = \frac{1}{k^2}$, [redacted] $k^2\sigma^2 \ge \lambda^2$

$$\text{Prob}\left(|X-\mu| > k\sigma\right) = \text{Prob}\left(|X-\mu|^2 > k^2\sigma^2\right)$$

$$\leq \text{Prob}\left(|X-\mu|^2 > \lambda^2\right) = \frac{1}{k^2} \quad [\text{QED}]$$

$$\langle W_N \rangle = \frac{1}{N} \Sigma \langle X_i \rangle = \mu$$

$$\text{Var}(W_N) = \frac{1}{N^2} \sum_i \text{Var}(X_i) = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$$

$$\text{Prob}\left(|W_N - \mu| > \frac{k\sigma}{\sqrt{N}}\right) \leq \frac{1}{k^2}$$

Put $k = \frac{\varepsilon\sqrt{N}}{\sigma}$. Then Chebyshev says

$$\text{Prob}\left(|W_N - \mu| > \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2 N}$$, which goes to zero as $N \to \infty$

Proof of Central limit thm. utilizes char. fncs.

$\boxed{\langle e^{itx} \rangle = F(t)}$

Note that if we do a $X \to X+a$ $F(k) \to e^{ika}F(k)$

$\langle e^{ik(x+a)} \rangle = e^{ika}\langle e^{ikx} \rangle = e^{ika}F(k)$

$\langle e^{ik(\lambda x)} \rangle = \langle e^{i(\lambda k)x} \rangle = F(\lambda k)$

$X \to \lambda X$ $F(k) \to F(\lambda k)$

Char fnc. of the distr'n $Z_N = \frac{W_N - \mu}{\sigma/\sqrt{N}}$

Note other way to say it is that $\frac{X_i-\mu}{\sigma}$ has a char. fnc. like $\left(1-\frac{k^2}{2}+o(k^2)\right)+\dots$

$Z_N \Rightarrow \left(1-\frac{k^2}{2N}+o\left(\frac{k^2}{N}\right)\right)^N$

$$Z_N = \frac{\sum_i X_i - N\mu}{\sqrt{N}\,\sigma}$$

Char fnc of each X_i is $f(k)$

$$\sum X_i \Rightarrow (f(k))^N$$

$$\sum X_i - N\mu \Rightarrow e^{-iNk\mu} f(k)^N = e^{-\frac{k^2}{2}N\sigma^2 + O(k^{2+\epsilon})}$$

[redacted]

when

big O

$$f(k) = e^{ik\mu + [redacted] \frac{(ik)^2}{2}\sigma^2 + O(k^{2+\epsilon})}$$

[redacted]

$$\frac{\sum X_i - N\mu}{\sqrt{N}\,\sigma} \Rightarrow [redacted]\ e^{-\frac{k^2}{2}} + O\left(\frac{k^{2+\epsilon}}{N^{\epsilon}[redacted]}\right)$$

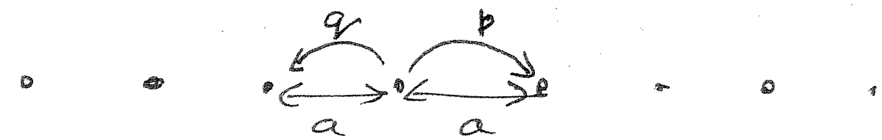
Vanishes when $N\to\infty$

[In fact all we needed was $\lim_{N\to\infty}\left(1-\frac{k^2}{2N}+o\left(\frac{k^2}{N}\right)\right)^N = e^{-k^2/}$[redacted]

small o

$$\lim_{N\to\infty} N\times o\left(\frac{k^2}{N}\right) = k^2 \lim_{N\to\infty}\frac{o\left(\frac{k^2}{N}\right)}{k^2/N} = k^2\times 0 = 0\ ]$$

We will discuss a stochastic processes in great detail later. But at this point we could consider particular Stochastic process in discrete time, namely, Random walk.



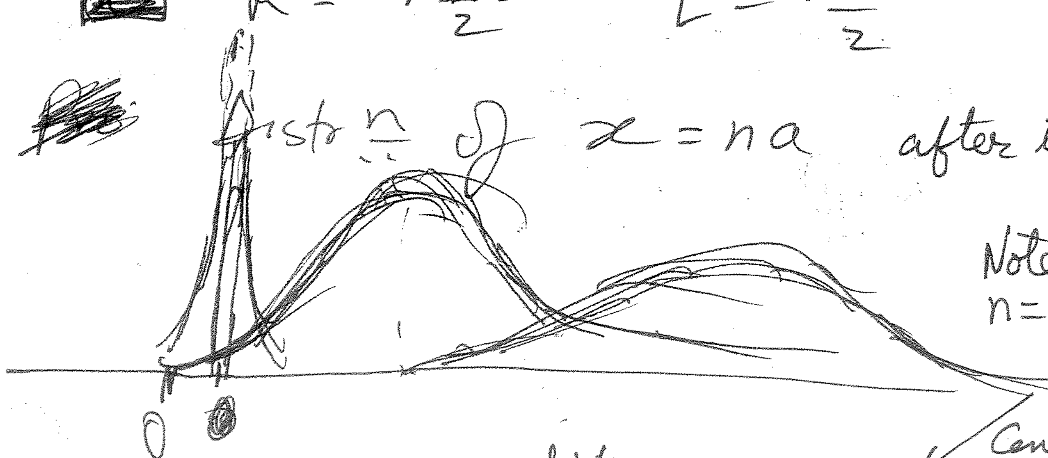
left moves L (Prob q)
Right " R (prob p)

$x = R - L$ = total displacement
$N = R + L$ = total # of steps

Distrⁿ of R $\quad Prob(R) = {}^{N}C_{R}\, p^{R} q^{L}$

$R = \frac{N+x}{2} \qquad L = \frac{N-x}{2}$

Distrⁿ of $x = na$ after time $t = N\tau$



Note, that
$n = \sum_{i=1} d_i$
$d_i = \pm 1$
with prob p.

Central Limit thm.

If N is large, x distr is approximately described by a gaussian with mean $= vt$ and variance $= 2Dt$, where $v = \frac{(p-q)a}{\tau}$, $D = \frac{2pqa}{\tau}$

$$P(x,t) \approx \frac{e^{-\frac{(x-vt)^2}{4Dt}}}{\sqrt{4\pi D t}}$$

D is the diffusion const, and v is the average drift.

Examples: Brownian motion in presence of a force [redacted], [redacted] carriers in a [redacted] semiconductor under a field.

There are generalizns of Central limit thm to distrn without well defined variance. We will leave that to an [redacted] exercise

We finish the prob. theory discussion by discussing large deviations in sum of random variables.

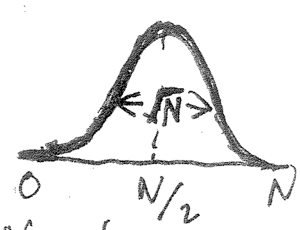
Let us take a concrete example: binomial distrn at $p = q = \frac{1}{2}$

[redacted] $P(n) = {}^{N}C_n \frac{1}{2^N}$

$n = \sum_{i=1}^{N} x_i$ where $x_i = 0, 1$ with prob $1/2$ each

x_i's are i.i.d

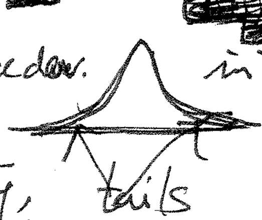
$P(n)$ peaks around $n = N/2$



We also know that for $n = N/2 + \frac{1}{2}\sqrt{N}\, z$

with z order 1, $P(n)$ is well approximated by a gaussian

$$\text{Const.}\; e^{-2(n-N/2)^2/N}$$ [redacted]

However we might need to know what happens when n deviates for $N/2$ by amounts order N and not order $\sqrt{N}$. Namely [redacted] $n =$ [redacted] N! with x order 1. We are probing the large dev.  in the tai

In this case we could, fortunately, explicitly calculate it, thanks to the Stirling formula

$$N! \simeq \sqrt{2\pi N}\, e^{-N} N^N$$

$$\frac{1}{2^N}\, {}^N C_n = \frac{1}{2^N} \frac{N!}{n!\,(N-n)!}$$ [redacted]

$$= \frac{1}{2^N} \frac{\sqrt{2\pi N}}{\sqrt{\pi N(1-x)}\sqrt{\pi N(1+x)}} \frac{e^{-N}}{e^{-N}} \frac{N^N}{\left[N\left(\frac{1+x}{2}\right)\right]^{N\left(\frac{1+x}{2}\right)} \left[N\left(\frac{1-x}{2}\right)\right]^{N\left(\frac{1-x}{2}\right)}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi N(1-x^2)}} \frac{1}{\left[(1+x)^{\frac{1+x}{2}}(1-x)^{\frac{1-x}{2}}\right]^N}$$

So $\ln P(n) = -N\left[\frac{1+x}{2}\ln(1+x) + \frac{1-x}{2}\ln(1-x)\right] + O(\ln N)$

Note that when x is small

$$\ln P(n) = -N\left[\frac{1+x}{2}\left(x - \frac{x^2}{2} + \cdots\right) + \frac{1-x}{2}\left(-x - \frac{x^2}{2} + \cdots\right)\right] + o(\ln N)$$

$$= -N\frac{x}{2}\left[(1+x)\left(1 - \frac{x}{2} + \cdots\right) - (1-x)\left(1 + \frac{x}{2} + \cdots\right)\right]$$

$$= -N\frac{x}{2} \times 2 \times \frac{1}{2}x = -\frac{Nx^2}{2}$$

So $P(n) \approx \frac{2}{\sqrt{\pi N}} e^{-N\frac{x^2}{2}}$

Note that in that limit $z = \frac{n - N/2}{\sqrt{N}/2} = \frac{N/2 \times x}{\sqrt{N}/2}$

$= [redacted] \sqrt{N}\, x$

So $P(n) \sim e^{-z^2/2}$ as [redacted] dictated by the central limit theorem. However for large x, we need $P(n) \approx \exp\left[-N\ [redacted]\ \phi(x)\right] / \sqrt{\pi N (1-x^2)/2}$

$$\phi(x) = \frac{1+x}{2}\ln(1+x) + \frac{1-x}{2}\ln(1-x)$$

[redacted]

Is there a general method to extract the distribution [redacted] of [redacted] large deviations [redacted] for sum of random variables? Yes, but the answer is dependent on the [redacted] details of the distribution of individual variables (beyond just the mean and variance, the only quantities that matter for the central limit theorem).

We first go the fancy route : [redacted] Do it by the saddle point method (evaluation of the characteristic func.

$$f(k) = \langle e^{ikx_i} \rangle = \tfrac{1}{2}(e^{ik}+1).$$

Characteristic func. of $n = \sum_i x_i$: $\left[\left(\frac{e^{ik}+1}{2}\right)\right]^N$

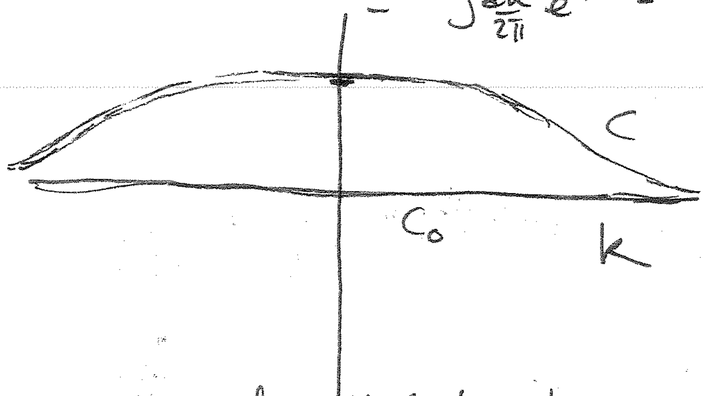
[redacted]

$$P(n) = \int \frac{dk}{2\pi} e^{-ikn} \left[\frac{e^{ik}+1}{2}\right]^N$$

Put $n = \frac{N}{2}(1+x)$

$$P(n) = \int \frac{dk}{2\pi} e^{-i[redacted]N\frac{kx}{2}} \left[\frac{e^{i\frac{k}{2}}+e^{-i\frac{k}{2}}}{2}\right]^N$$

$$= \int \frac{dk}{2\pi} e^{N[-ikx/2 + \ln\cos k/2]}$$



Could we do this k integral by some approximation [redacted] for N large

Let us find a point in compl k space where the integral is stationary

[redacted]

Expand $-ik\frac{x}{2} + \ln\cos k/2$

$$\boxed{\tan k/2 = -ix}$$

$\Rightarrow$ k is imaginary, $k = -i\kappa$

$$\text{th}\frac{\kappa}{2} = x \Rightarrow x = \frac{e^{\kappa/2} - e^{-\kappa/2}}{e^{\kappa/2}+e^{-\kappa/2}} \Rightarrow \frac{1+x}{1-x} = e^{\kappa}$$

$$k = -i\kappa + \delta k$$

$$-ikx/_2 + \ln \cos k/_2 = -\frac{1}{2}\kappa x + \ln\cosh\frac{\kappa}{2}$$
$$\cdot -\frac{1}{8}\,\mathrm{sech}^2\frac{\kappa}{2}\ \delta k^2$$

$$e^{N[\quad]} = e^{N\left(-\frac{1}{2}\kappa x + \ln \mathrm{ch}\frac{\kappa}{2}\right)}\ e^{-\frac{1}{8}\mathrm{sech}^2\frac{\kappa}{2}\delta k^2}$$

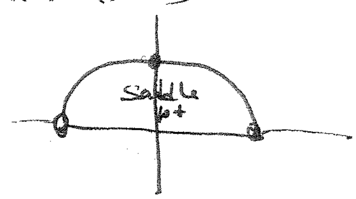
$$\bullet\ \mathrm{ch}\frac{\kappa}{2} = \frac{e^{\kappa/2}+e^{-\kappa/2}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)$$
$$= \frac{1}{2}\,\frac{1+x+1-x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$-\frac{1}{2}\kappa x + \ln \mathrm{ch}\frac{\kappa}{2} = [redacted]\, x \ln\sqrt{\frac{1-x}{1+x}} + \ln\frac{1}{\sqrt{1-x^2}}$$
$$= -\left[\frac{1+x}{2}\ln(1+x) + \frac{1-x}{2}\ln(1+x)\right]$$

$$\mathrm{sech}^2\frac{\kappa}{2} = (1-x^2)$$

$$\int \frac{d\delta k}{2\pi}\, e^{-N\frac{(1-x^2)}{8}\delta k^2} = \frac{\sqrt{\pi \times \frac{8}{1-x^2}\times\frac{1}{N}}}{2\pi} = \frac{\sqrt{2}}{\sqrt{\pi N(1-x^2)}}$$

BTW $\quad P(n) = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\, e^{-ikn}\left(\frac{e^{ik}+1}{N}\right)^N$



Poisson Summ^n

$\sum_n f(x+n) = \Phi(x)$ Periodic

$\int_0^1 \sum_n f(x+n)\, e^{2\pi m x i}\, dx$

$= \int_{-\infty}^{\infty} f(x)\, e^{2\pi m x i}\, dx$

$= \hat{f}(2\pi m)$

$\Phi(x) = [redacted] \sum_m \hat{f}(2\pi m) e^{2\pi m x i}$

for $x = 0$

$\sum f(n) = [redacted] \sum_m \hat{f}(2\pi m)$

Formal application $\quad \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{-iky}\left(\frac{e^{ik}+1}{2}\right)^N = \sum_n P(y)\,\delta(y-n)$

$$\int_{-\infty}^{\infty} e^{-iky}\left(\frac{e^{ik}+1}{2}\right)^N \frac{dk}{2\pi} = \sum_m \int_{(2m-1)\pi}^{(2m+1)\pi} e^{-iky}\left(\frac{e^{ik}+1}{2}\right)^N \frac{dk}{2\pi} = [redacted]\underbrace{\left(\sum_m e^{2m\pi y i}\right)}_{\sum_n \delta(y-n)}\ \underbrace{[redacted]\int_{-\pi}^{\pi} e^{-iky}\left(\frac{e^{ik}+1}{2}\right)^N \frac{dk}{2\pi}}_{P(y)}$$

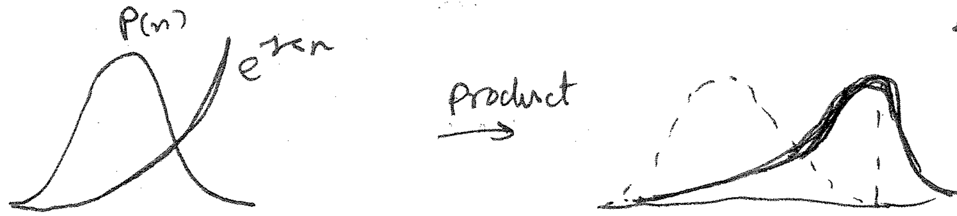
$\delta(x-y) \xrightarrow{FFT} \int dx\, e^{ikx}\delta(x-y) = e^{iky}$

Poisson Summation formula

The third way to get the same answer: Consider the characteristic func with $ik = \kappa$

$$f(-i\kappa) = \sum_n P(n) e^{\kappa n} = Z(\kappa)$$

Consider now a new prob distr^n: $\tilde{P}(n) = \dfrac{e^{\kappa n} P(n)}{Z(\kappa)}$



max position defines $\bar{n}$

$$\tilde{P}(n) \sim \text{Prefactor } e^{\frac{N\kappa x}{2}} e^{-N\phi(x)}$$

$$\approx \text{Prefactor } e^{N(\kappa\bar{x} - \phi(\bar{x}))} e^{-\frac{1}{2}N\phi''(\bar{x})(x-\bar{x})^2}$$

$$\langle \delta x^2 \rangle \sim \frac{1}{N} \qquad (\delta n)^2 = \left\langle \left(\frac{N\delta x}{2}\right)^2 \right\rangle \sim N$$

$$n \sim N \pm \sqrt{N}$$

Find κ, s.t. $\langle n \rangle = \bar{n}$

$$\bar{n} = \sum_n n \tilde{P}(n) = \frac{\sum n e^{\kappa n} P(n)}{\sum e^{\kappa n} P(n)} = \partial_\kappa \ln \sum e^{\kappa n} P(n)$$

$$Z(\kappa) = \sum_n P(n) e^{\kappa n} \simeq P(\bar{n}) e^{\kappa\bar{n}} \sum_n e^{-\alpha \delta n^2 / 2N} = P(\bar{n}) e^{\kappa\bar{n}} \times \sqrt{\frac{2\pi N}{\alpha}}$$

Note that $\partial_\kappa^2 \ln Z(\kappa) = \partial_\kappa \dfrac{\sum_n n e^{\kappa n} P(n)}{\sum e^{\kappa n} P(n)}$

$$= \frac{\sum n^2 e^{\kappa n} P(n)}{\sum e^{\kappa n} P(n)} - \left(\frac{\sum n e^{\kappa n} P(n)}{\sum e^{\kappa n} P(n)} \right)^2 = \langle (n - \bar{n})^2 \rangle_{\tilde{P}} = \frac{N}{\alpha}$$

$$Z(\kappa) = \left(\frac{1+e^{\kappa}}{2}\right)^N = e^{\frac{\kappa N}{2}} \operatorname{ch}^N \frac{\kappa}{2}$$

$$\partial_\kappa \ln Z(\kappa) = \frac{N}{2} + \frac{N}{2} \operatorname{th} \frac{\kappa}{2}$$

$$\bar{n} = \frac{N}{2}\left(1 + \operatorname{th} \frac{\kappa}{2}\right) \Rightarrow \frac{x}{2} = \operatorname{th} \frac{\kappa}{2}$$

$$P(\bar{n}) = e^{-\kappa \bar{n}} Z(\kappa) \sqrt{\frac{\alpha}{2\pi N}}$$

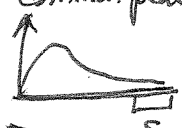
$$= e^{\kappa(N-\bar{n})} \operatorname{ch}^N \frac{\kappa}{2} \sqrt{\frac{\alpha}{2\pi N}} = e^{\kappa(N-\bar{n})} \operatorname{ch}^N \frac{\kappa}{2} \times \sqrt{\frac{2}{\pi \operatorname{sech}^2 \frac{\kappa}{2}}}$$

$$\frac{N}{\alpha} = \partial_\kappa^2 \ln Z(\kappa) = \frac{N}{4} \operatorname{sech}^2 \frac{\kappa}{2}$$

$$P(\bar{n}) = e^{-N\Phi(x)} \sqrt{\frac{2}{\pi(1-x^2)N}}$$

Forcing $P(n)$ to $\hat{P}(n)$ by multiplying with a factor like e^{xn}; making something [redacted] rare into something commonplace is a very useful trick in simulations

$$\langle 1_s(n) \rangle_P = \sum_n 1_s(n) P(n) = \frac{\sum_n 1_s(n) e^{-xn} e^{xn} P(n)}{\sum_n e^{-xn} e^{xn} P(n)} = \frac{\langle 1_s(n) e^{-xn} \rangle_{\hat{P}}}{\langle e^{-xn} \rangle_{\hat{P}}}$$



importance sampling

Better noise properties