Course: Statistical Mechanics
Problem Set: 3
Due: 27th Apr 2007

1. Massless Bosons in Two Dimensions: Let us consider bosons in two dimensions with the energy for the particle with wave number $\vec{k}$ being given by

$$\epsilon_{\vec{k}} = \hbar\omega_{\vec{k}} = \hbar v|\vec{k}|$$

where v is the speed of the particles. Would there be a Bose-Einstein condensate in this system and, if so, under what conditions (in terms of temperature and number density) would it form?

2. Relativistic Fermions at High Densities: The core of collapsed stars are often stabilized by the pressure of fermions (electrons, neutrons,..) at high density. Remember that the relativistic energy momentum relation is given by

$$\epsilon^2 = m^2c^4 + p^2c^2$$

(ϵ, m, c, p being the energy, the mass, the velocity of light and the momentum, respectively) and that $p = \hbar k$ according to quantum mechanics where $k = |\vec{k}|$ is the wave number. Let $\rho = N/V$ be the number density (we have N particles in volume V). Imagine that, for a fixed temperature, we let ρ get very large. In this limit there is an asymptotic relation between the pressure P and density ρ. Find this relation.

(Hint: For very high densities, most fermions would occupy very high values of k, because of Pauli principle. As a result, for most of them, $pc >> mc^2$, allowing you to neglect the mass.)

3. Mean Field Theory of Phase Transition in Potts Model: In Ising model, spins have two values, up and down. If neighboring spins are the same, we get an energy $-J$ and if they are not the same, we get $+J$. When J is positive, we have ferromagnetic interaction. The energy penalty for the spins being different in neighboring sites is $2J = (+J) - (-J)$. We found that, in the mean field theory, we have a phase transition at lower temperature where all the spins are more likely to be up (or down). If we define probability of being up(down) $p_{\uparrow,\downarrow} = < N_{\uparrow,\downarrow} > /N$, N being the total number of spins, and $N_{\uparrow,\downarrow}$ is the number of up (down) spins in a particular spin configuration, then the magnetization per spin is

$$m = p_\uparrow - p_\downarrow.$$

Our mean field equation for m is given by

$$m = \frac{e^{2\beta zJm} - 1}{e^{2\beta zJm} + 1} = \tanh(\beta zJm)$$

where z is the number of neighbors of each site in the lattice. We found that, at low temperatures, there is a solution with $m \neq 0$.

The Potts model is a generalization of Ising model where spins can take q values: $\sigma = 1, 2, 3, .., q$. The energy penalty of not having the same value of σ on neighboring sites is $2J$ (we keep $J > 0$, making this the ferromagnetic Potts model). We formalize this by defining the total energy of the system as

$$E = -2J \sum_{<ij>} \delta_{\sigma_i \sigma_j},$$

with $< ij >$ indicating that we sum over pairs which are nearest neighbors. The Kronecker delta, $\delta_{ab} = 1$ if $a = b$ and it is zero otherwise. The lowest energy state correspond to all spins having the same value (which could be, say, for all i, $\sigma_i = 1$). There are q such states, depending which of the q possible spin values were chosen by every spin.

At high temperatures, the entropy gain from each spin fluctuating equally among all the q possible values would favor the paramagnetic state. In this state, the probability of the spin being in any state $\sigma = 1, .., q$, which is $p_\sigma =< N_\sigma > /N$ is just $1/q$. However, at lower temperatures, it is, in principle, possible to get symmetry broken states where, say, p_1 is different from $p_2 = p_3 = \cdots = p_q$. In analogy with the Ising model, we define

$$m = p_1 - p_\sigma,$$

with $\sigma \neq 1$, for such a state.

In the following steps, you will use the variational principle to derive the mean field equation for the Potts model, with

$$E_0 = -2h_{eff} \sum_i \delta_{\sigma_i,1}.$$

Assume that on the lattice, every site has z neighbors.

1) Find the expression for m in terms of h_{eff} in the thermal ensemble defined by E_0 (calculate p_σ with the formula $< N_\sigma >_0 /N$, $<>_0$ referring to thermal average in the ensemble defined by E_0).

2) Compute

$$\Phi =< E >_0 -TS_0$$

and express it as a function of m.

3) Minimize $\Phi(m)$ to find the mean field equation for m.

4) On a three dimensional cubic lattice, the number of nearest sites for every site is 6. Let us take the Potts model with $q = 3$ on such a lattice. What is the value of $K = \beta J$, above which we will have an ordered state with $m \neq 0$? Find the critical value K_c numerically. As K passes though K_c, do we have a first order transition or a second order transition?