Course: Statistical Mechanics
Problem Set: 1
Due: 23rd Feb 2007

1. Ideal Gas Entropy: We know that the equation of state of ideal gas is

$$PV = nRT,$$

(usual notation, n being the, the number of moles) and that

$$P = \frac{2U}{3V}$$

from kinetic theory. Calculate, the dependence of entropy on U and V, namely, $S = S(U,V)$, up to an additive constant, for a fixed number of particles, $N = nN_A$, (N_A being the Avogadro number).

Given that entropy is extensive, determine, the N dependence, in addition to the U, V dependence. In other word, find the function $S = S(U,V,N)$ for ideal gas up to additive terms of the form constant$\times N$.

Change variables and express, up to an additive constant independent of N, entropy per particle, S/N, in terms of temperature T and particle density $\rho = N/V$.

2. Gambling: Let us say I offer you the following bet. We throw a pair of (unbiased) dice and getting a "double 6" is a success. I let you have 24 efforts at it. If you do not succeed in any of these 24 trials, you give me a dollar. If any of your 24 "double throws" is a success (whether you hit one or more does not matter), you get a dollar. On the average, who is going to make money in this game, you or me?

What would your conclusion be, if the number of trials increases to 25?

3. Product of Random Variables: Imagine a random process in which , at every step, with equal probability, either the variable get scaled up by e^θ or get scaled down, being multiplied by $e^{-\theta}$. More precisely, $X_0 = 1$

$$\begin{aligned} X_{n+1} &= e^{\theta} X_n, \text{ with probability } \frac{1}{2} \\ X_{n+1} &= e^{-\theta} X_n, \text{ with probability } \frac{1}{2} \end{aligned}$$

a) Calculate the average value of X_N (I will denote it by $< X_N >$).

b) What is the asymptotic distribution of X_N for large N. [Hint: A combination logarithmic transformation and central limit theorem might help.]

4. Markov Chain: Here is process with transition matrix

$$Q = \begin{pmatrix} 0 & q & p \\ p & 0 & q \\ q & p & 0 \end{pmatrix}$$

with $p + q = 1$ It represents hopping on a triangle with different probabilities for left hop and for right hop. Initially, the system is in state one, meaning the probability vector is given by

$$\mathbf{P}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Compute $\mathbf{P}(s) = Q^s \mathbf{P}(0)$.

5. Birth-Death Process: The rate at which an object is created is r. The rate at which any of the current n objects could disappear is $1/\tau$ (τ is the average life time of each of these objects). Calculate the stationary distribution of n. [Hint: Show that detailed balance holds here and use it to solve $P(n)$, the probability of having n objects in the stationary state.]

6. Cauchy Distribution: The probability density function of a continuous random variable X is

$$p_1(X) = \frac{a}{\pi} \frac{1}{X^2 + a^2}$$

This is a Cauchy distribution.

a) Calculate its characteristic function $F(k) = < e^{ikX} >$. Can you expand this function around $k = 0$ to get the moments? Why not?

b) If Y is distributed according to a different Cauchy distribution

$$p_2(Y) = \frac{b}{\pi} \frac{1}{Y^2 + b^2}$$

and X and Y are independent. Find the probability density function of $Z = X + Y$.