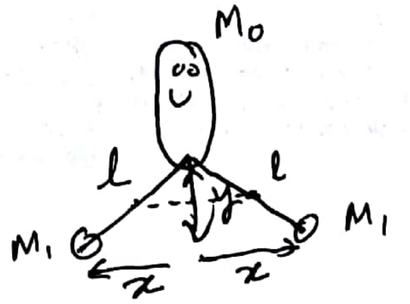
2018 Classical Mechanics Midterm Answers

1. a)



$$x = \sqrt{l^2 - y^2}$$

$$\xi = x$$

$$T = \frac{1}{2} M_0 \dot{y}^2 + 2 \frac{1}{2} m_1 \dot{x}^2 , \quad V = \frac{1}{2} k \xi^2 + M_0 g y$$

$$T = \frac{1}{2} M_0 \dot{y}^2 + m_1 \left(\frac{d\sqrt{l^2-y^2}}{dy} \dot{y} \right)^2$$

$$= \frac{1}{2} M_0 \dot{y}^2 + m_1 \left(\frac{1(-2y)}{2\sqrt{l^2-y^2}} \dot{y} \right)^2$$

$$= \frac{1}{2} \left\{ M_0 + 2m_1 \frac{y^2}{l^2-y^2} \right\} \dot{y}^2$$

$$= \frac{M_0 l^2 + (2m_1 - M_0) y^2}{2(l^2-y^2)} \dot{y}^2$$

$$V = \frac{1}{2} k \xi^2 + M_0 g y = \frac{1}{2} k (l^2 - y^2) + m_0 g y$$

$$L = T - V = \frac{1}{2} \left\{ \frac{M_0 l^2 + (2m_1 - M_0) y^2}{l^2 - y^2} \right\} \dot{y}^2 - \frac{1}{2} k (l^2 - y^2) - m_0 g y$$

Define $M(y) = \frac{M_0 l^2 + (2m_1 - m_0) y^2}{l^2 - y^2} = \frac{2m_1 l^2}{l^2 - y^2} + m_0 - 2m_1$

$$L = \frac{1}{2} M(y) \dot{y}^2 - \frac{1}{2} k (l^2 - y^2) - m_0 g y$$

b) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$

$$\frac{d}{dt}\left[M(y)\dot{y}\right] = \frac{1}{2}\frac{\partial M(y)}{\partial y}\dot{y}^2 + ky - m_0 g$$

$$M(y)\ddot{y} + \frac{\partial M}{\partial y}\dot{y}^2 = \frac{1}{2}\frac{\partial M}{\partial y}\dot{y}^2 + ky - m_0 g$$

$$M(y)\ddot{y} + \frac{1}{2}\frac{\partial M}{\partial y}\dot{y}^2 = ky - m_0 g$$

$$\frac{\partial}{\partial y}M(y) = \frac{\partial}{\partial y}\left(\frac{2m_1 l^2}{l^2 - y^2} + m_0 - 2m_1\right) = -\frac{2m_1 l^2(-2y)}{(l^2 - y^2)^2} = \frac{4m_1 l^2 y}{(l^2 - y^2)^2}$$

So $$\frac{m_0 l^2 + (2m_1 - m_0)y^2}{l^2 - y^2} + \frac{2m_1 l^2 y}{(l^2 - y^2)^2}\dot{y}^2 = ky - m_0 g$$

Scanned by CamScanner

2. a) $\mathcal{L} = \frac{m}{2}\left(\frac{\dot{r}^2}{1-\frac{a}{r}} + r^2\dot{\theta}^2\right) + \frac{2\lambda}{\left(1-\frac{a}{r}\right)r}$

No explicit θ (rotational invariance)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = l \quad \text{conserved}$$

No explicit t (time independence)

$$H = \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = \frac{m}{2}\left(\frac{\dot{r}^2}{1-\frac{a}{r}} + r^2\dot{\theta}^2\right) - \frac{2\lambda}{\left(1-\frac{a}{r}\right)r} = E$$

conserved

b) $$\frac{m}{2}\left[\frac{\dot{r}^2}{\left(1-\frac{a}{r}\right)} + r^2\dot{\theta}^2\right] - \frac{2\lambda}{\left(1-\frac{a}{r}\right)r} = E$$

Eliminate $\dot{\theta}$ using $\dot{\theta} = \frac{l}{mr^2}$

$$\frac{m}{2}\left[\frac{\dot{r}^2}{1-\frac{a}{r}} + \frac{l^2}{m^2 r^2}\right] - \frac{2\lambda}{\left(1-\frac{a}{r}\right)r} = E$$

Now multiply by $1-\frac{a}{r}$.

$$\frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2}\left(1-\frac{a}{r}\right) - \frac{2\lambda}{r} = E\left(1-\frac{a}{r}\right)$$

$$\frac{m}{2}\dot{r}^2 - \frac{2\lambda - Ea}{r} + \frac{l^2}{2mr^2} - \frac{l^2 a}{2mr^3} = E$$

So $$V'(r) = -\frac{2\lambda - Ea}{r} + \frac{l^2}{2mr^2} - \frac{l^2 a}{2mr^3} = -\frac{A}{r} + \frac{B}{r^2} - \frac{C}{r^3}$$

c) With $a = 0$

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{2\lambda}{r}$$

This is just Kepler problem with the attactive potential $-\frac{2\lambda}{r}$.

We will have elliptic closed orbits and parabolic and hyperbolic orbits for the particle escaping to infinity.

d) We will consider the effect of nonzero a on the elliptic orbits. We have seen the closed elliptic orbits are related to the frequency of angular motion matching the frequency of small oscillations in r around a circular orbit. We will ~~approximately~~ calculate the small mismatch caused by nonzero a, when $a \ll r$.

To keep notation close to the Kepler case,
we will call $2\lambda - Ea = k$

$$V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2} - \frac{l^2 a}{2mr^3}$$

The angular motion is given by
$\dot{\theta} = \frac{l}{mr^2}$. For a circular orbit $r = r_{eq}$,

$\dot{\theta} = \frac{l}{mr_{eq}^2} = \omega_0$, the frequency
of angular motion for circular orbit.

The radial motion equation is

$$m\ddot{r} = -\frac{d}{dr} V'(r)$$

The circular orbit, $r = r_{eq}$ satisfies $\quad -\frac{d}{dr} V'(r)\Big|_{r=r_{eq}} = 0$

Small perturbation $\quad r(t) = r_{eq} + \delta r(t)$

satisfies $\quad m\ddot{\delta r} = -\left[\frac{d^2}{dr^2} V'(r)\Big|_{r=r_{eq}}\right] \delta r$

$$= -m\omega_r^2 \delta r$$

$\delta r(t) \simeq A\cos\omega_r t \quad$ with $\omega_r^2 = \frac{1}{m}\frac{d^2}{dr^2} V'(r)\Big|_{r=r_{eq}}$

For us

$$\frac{d}{dr}V'(r) = \frac{k}{r^2} - \frac{l^2}{mr^3} + \frac{3l^2a}{2mr^4}$$

$$\frac{1}{m}\frac{d^2}{dr^2}V'(r) = -\frac{2k}{mr^3} + \frac{3l^2}{m^2r^4} - \frac{6l^2a}{m^2r^5}$$

$r=r_{eq}$ satisfies $\dfrac{k}{r_{eq}^2} = \dfrac{l^2}{mr_{eq}^3} - \dfrac{3l^2a}{2mr_{eq}^4}$

Using that in $\frac{1}{m}\frac{d^2}{dr^2}V'(r)$

$$\omega_r^2 = -\frac{2}{mr_{eq}}\left\{\frac{l^2}{mr_{eq}} - \frac{3l^2a}{2mr_{eq}^4}\right\} + \frac{3l^2}{m^2r_{eq}^4} - \frac{6l^2a}{m^2r_{eq}^5}$$

$$= \frac{l^2}{m^2r_{eq}^4} - \frac{3l^2}{m^2r_{eq}^5}\frac{a}{r_{eq}}$$

$$= \omega_0^2\left(1 - \frac{3a}{r_{eq}}\right)$$

Now things are slightly mismatched. The radial oscillation is slower. When radius returns to the same value (say minimum), θ has moved past 2π.

The "ellipse" rotates!

3. The particle locations are

$(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,a)$

$(0,\frac{a}{2},\frac{a}{2})$, $(\frac{a}{2},0,\frac{a}{2})$, $(\frac{a}{2},\frac{a}{2},0)$.

a) Since x,y,z permutation is a symmetry it is enough to compute I_{11} and I_{12}

$$I_{11} = \sum_i m_i(y_i^2+z_i^2) \qquad I_{12} = -\sum_i m_i x_i y_i$$

$$I_{11} = m\left[0+0+a^2+a^2+\left(\frac{a^2}{4}+\frac{a^2}{4}\right)+\frac{a^2}{4}+\frac{a^2}{4}\right]$$

$$= 3ma^2$$

$$I_{12} = -m\left[0+0+0+0+\frac{a^2}{4}+0+0\right]$$

So $I = ma^2\begin{bmatrix} 3 & -1/4 & -1/4 \\ -1/4 & 3 & -1/4 \\ -1/4 & -1/4 & 3 \end{bmatrix}$

b) $I = ma^2\left[\left(3+\frac{1}{4}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\left(-\frac{1}{4}\right)\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right]$

The matrix $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ has eigenvector $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$ with

Scanned by CamScanner

eigenvalue 3, with two other eigenvectors which are orthogonal to $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, which have eigenvalue zero.

We could choose.

$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$$\underline{\underline{I}}\, u_1 = ma^2\left\{\left(3+\frac{1}{4}\right)u_1 + \left(-\frac{1}{4}\right)3u_1\right\} = \frac{5}{2}ma^2 u_1$$

$$\underline{\underline{I}}\, u_{2,3} = ma^2\left\{\left(3+\frac{1}{4}\right)u_{2,3} + 0\right\} = \frac{13}{4}ma^2 u_{2,3}$$

So the principal moments of inertia are $\frac{5}{2}ma^2$, $\frac{13}{4}ma^2$, $\frac{13}{9}ma^2$. The principal axis for the first one is along $\begin{pmatrix}1\\1\\1\end{pmatrix}$ vector. The other two are any two orthogonal axes perpendicular to $\begin{pmatrix}1\\1\\1\end{pmatrix}$

4. Apologies! It should have been:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

a) $$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega}_r \times \vec{L}$$

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega}_r \times \vec{L} = \vec{N}$$

b) $$I_1 \dot{\omega}_1' + \omega_{r2}' I_3 \omega_3' - \omega_{r3}' I_1 \omega_2' = N_1'$$

$$I_2 \dot{\omega}_2' + \omega_{r3}' I_1 \omega_1' - \omega_{r1}' I_3 \omega_3' = N_2'$$

$$I_3 \dot{\omega}_3' + \omega_{r1}' I_1 \omega_2' - \omega_{r2}' I_1 \omega_1' = N_3'$$

c) $\vec{N}$ in x', y', z' is $(Mgl\sin\theta, 0, 0)$

d)
$$\omega_1' = \dot{\theta} \qquad \omega_{r1}' = \dot{\theta}$$
$$\omega_2' = \dot{\phi}\sin\theta \qquad \omega_{r2}' = \dot{\phi}\sin\theta$$
$$\omega_3' = \dot{\psi} + \dot{\phi}\cos\theta \qquad \omega_{r3}' = \dot{\phi}\cos\theta$$

Notice that $\omega_{r1}\omega_2' - \omega_{r2}\omega_1' = \omega_1'\omega_2' - \omega_2'\omega_1' = 0$

$\Rightarrow \dot{\omega}_3' = 0$

e) In steady condition $\dot{\omega}_i = 0$

$$I_1\dot{\omega}_1' + \omega_{r2} I_3 \omega_3' - \omega_{r3} I_1 \omega_2' = N$$

$$\Rightarrow \quad 0 + \dot{\phi}\sin\theta\, I_3 \omega_3' - \dot{\phi}\cos\theta\, I_1 \dot{\phi}\sin\theta = mgl\sin\theta$$

Taking out the common $\sin\theta$ factor

$$Mgl = \dot{\phi}\left(I_3\omega_3' - \dot{\phi}\cos\theta\right)$$