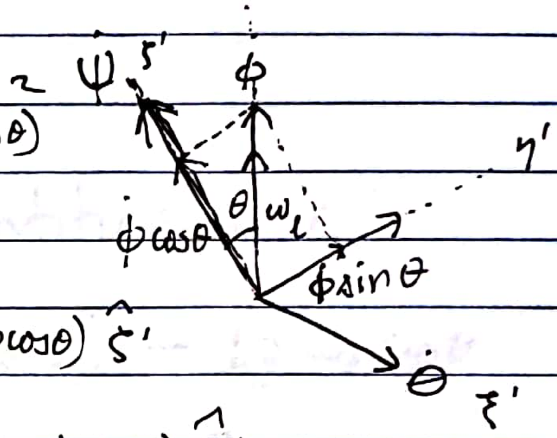
Ch 5, Prob 28

Choose the $z$ axis to line with $\vec{\omega}_e$, and use Euler angles




$$T = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2$$

$$\vec{\omega} = \dot\theta\,\hat{\xi}' + \dot\phi\sin\theta\,\hat{\eta}' + (\dot\psi + \dot\phi\cos\theta)\,\hat{\zeta}'$$

$$\vec{L} = I_1(\dot\theta\,\hat{\xi}' + \dot\phi\sin\theta\,\hat{\eta}') + I_3(\dot\psi + \dot\phi\cos\theta)\,\hat{\zeta}'$$

$$\vec{\omega}_e = \omega_e(\sin\theta\,\hat{\eta}' + \cos\theta\,\hat{\zeta}')$$

So $$\vec{\omega}_e \cdot \vec{L} = \omega_e\left[ I_1 \dot\phi \sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta \right]$$

The Lagrangian $\mathcal{L} = T - \vec{\omega}_e \cdot \vec{L}$ has no explicit dependence on time, $\psi$ or $\phi$. The first conserved quantity is therefore the Hamiltonian $h = p_i \dot q_i - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot q_i} \dot q_i - \mathcal{L}$

(using summation convention)

Note that $\vec{\omega} \cdot \vec{L}$ is linear in $\dot q_i$ and such terms get cancelled between $\frac{\partial \mathcal{L}}{\partial \dot q_i} \dot q_i$ and $\mathcal{L}$.

Only terms that contribute is $T$

$$h = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L} = \left(2T - \vec{\omega}_e \cdot \vec{L}\right) - \left(T - \vec{\omega}_e \cdot \vec{L}\right) = T$$

which is conserved.

Other two conserved quantities are:

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) - I_3\omega_e\cos\theta$$

$$= I_3\left[\dot{\psi} + (\dot{\phi} - \omega_e)\cos\theta\right]$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

$$- I_1\omega_e\sin^2\theta - I_3\omega_e\cos^2\theta$$

$$= I_1(\dot{\phi} - \omega_e)\sin^2\theta + I_3\{\dot{\psi} + (\dot{\phi} - \omega_e)\cos\theta\}\cos\theta$$

Note that $p_\phi, p_\psi$ are essentially the same expression as $\omega_e = 0$ but with $\dot{\phi} \to \dot{\phi} - \omega_e$

Using that we can rewrite the Lagrangian this way.

$$\mathcal{L} = \frac{1}{2} I_1\left[\dot{\theta}^2 + (\dot{\phi} - \omega_e)^2\sin^2\theta\right] + \frac{1}{2}I_3\left[\dot{\psi} + (\dot{\phi} - \omega_e)\cos\theta\right]^2$$

$$- \frac{1}{2}I_1\omega_e^2\sin^2\theta - \frac{1}{2}I_3\omega_e^2\cos^2\theta$$

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Essentially we can go to rotating frame around $z$ axis, with $\tilde{\phi} = \phi - \omega_e t$

$$\mathcal{L} = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\tilde{\phi}}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\tilde{\phi}} \cos\theta\right)^2$$

$$- \frac{1}{2} I_1 \omega_e^2 (1 - \cos^2\theta) - \frac{1}{2} I_3 \omega_e^2 \cos\theta_3$$

$$= \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\tilde{\phi}}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\tilde{\phi}} \cos\theta\right)^2$$

$$+ \frac{1}{2}(I_1 - I_3)\omega_e^2 \cos^2\theta \underbrace{- \frac{1}{2} I_1 \omega_e^2}_{\text{const}}$$

$\frac{1}{2}(I_1 - I_3)\omega_e^2 \cos^2\theta$ replaces the term due to gravity.

Now let us play the same game as for the heavy top.

$$p_\psi = I_3 \dot{\psi} + \dot{\tilde{\phi}} \cos\theta = I_1 a, \quad p_\phi = I_1 \dot{\tilde{\phi}} \sin^2\theta + I_3 (\dot{\psi} + \dot{\tilde{\phi}} \cos\theta) \cos\theta$$

$$= I_1 b$$

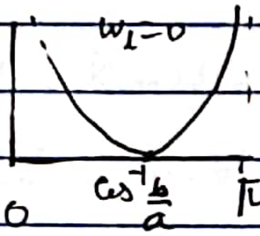
Call $\dot{\psi} + \dot{\tilde{\phi}} \cos\theta = \tilde{\omega}_3$. Note that $\dot{\tilde{\phi}} = \frac{b - a\cos\theta}{\sin^2\theta}$

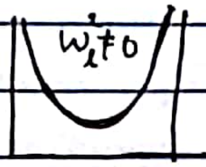
$$E' = E - \frac{1}{2} I_3 \tilde{\omega}_3^2 - \frac{1}{2} I_1 \omega_e^2 = \frac{1}{2} I_1 \left[\dot{\theta}^2 + \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2\right]$$

$$+ \frac{1}{2}(I_1 - I_3)\omega_e^2 \cos^2\theta$$

Let us look at the effective potential in the limit $a, b \gg \omega_e$

$$V_{eff}(\theta) = \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \frac{1}{2}(I_1 - I_3)\, \omega_e^2 \cos^2\theta$$




$V_{eff}(\theta)$ minimizes at $\theta_{eq}$ $\quad \left.\frac{d}{d\theta}V(\theta)\right|_{\theta=\theta_{eq}} = 0$

$\tilde{\dot{\phi}} = \frac{b - a\cos\theta_{eq}}{\sin^2\theta_{eq}}$

Also nutation freq $\left.\sqrt{\frac{1}{I_1}\frac{d^2V(\theta)}{d\theta^2}}\right|_{\theta=\theta_{eq}}$

At $\omega_\ell = 0$ $\quad b - a\cos\theta_{eq} = 0$

$$\frac{d^2}{d\theta^2}V(\theta) = I_1 \left(\frac{a\sin\theta}{\sin^2\theta}\right)^2 + o(b - a\cos\theta) + o(\omega_\ell^2)$$

To lead order, nutation frequency is $a$.

So precession $\dot{\phi} = \omega_\ell + \tilde{\dot{\phi}}$

$= \omega_\ell + o(\omega_\ell^2)$

The spin around the figure axis is $\Omega$, say. $\tilde{\omega}_3 \approx \Omega$.

$a = \frac{I_3\tilde{\omega}_3}{I_1} \approx \frac{I_3\Omega}{I_1}$ = nutation freq. to leading order.

The kinetic energy is conserved, So all this oscillation comes from different modes of rotation exchanging energies.

Ch 6, Prob 6a

$$K.E. = \frac{1}{2}m\left(\dot{\eta}_1^2 + \dot{\eta}_5^2 + \dot{\eta}_3^2\right) + \frac{1}{2}M\left(\dot{\eta}_2^2 + \dot{\eta}_4^2\right)$$

$$P.E. = \frac{1}{2}k\left[(\eta_1-\eta_2)^2 + (\eta_2-\eta_3)^2 + (\eta_3-\eta_4)^2 + (\eta_4-\eta_5)^2\right]$$

$$= \frac{k}{2}\left[\eta_1^2+\eta_5^2 - 2(\eta_1\eta_2+\eta_5\eta_4) + 2(\eta_2^2+\eta_4^2) - 2(\eta_2+\eta_4)\eta_3 + 2\eta_3^2\right]$$

Define $\eta_3 = \zeta_3$, $\eta_1 = \frac{\zeta_1+\zeta_5}{\sqrt{2}}$, $\eta_5 = \frac{\zeta_1-\zeta_5}{\sqrt{2}}$, $\eta_2 = \frac{\zeta_2+\zeta_4}{\sqrt{2}}$, $\eta_4 = \frac{\zeta_2-\zeta_4}{\sqrt{2}}$

Using these,

$$K.E. = \frac{m}{2}\left(\dot{\zeta}_1^2 + \dot{\zeta}_5^2 + \dot{\zeta}_3^2\right) + \frac{M}{2}\left(\dot{\zeta}_2^2 + \dot{\zeta}_4^2\right)$$

$$P.E. = \frac{k}{2}\left[\zeta_1^2+\zeta_5^2 - 2(\zeta_1\zeta_2+\zeta_4\zeta_5) + 2(\zeta_2^2+\zeta_4^2) - 2\sqrt{2}\,\zeta_2\zeta_3 + 2\zeta_3^2\right]$$

Notice that Lagrangian breaks up in two decoupled parts

$$\mathcal{L} = \left[ \frac{m}{2}\left(\dot{\zeta}_1^2 + \dot{\zeta}_3^2\right) + \frac{M}{2}\dot{\zeta}_2^2 - \frac{k}{2}\left\{ \zeta_1^2 - 2\zeta_1\zeta_2 + 2\zeta_2^2 - 2\sqrt{2}\,\zeta_2\zeta_3 + 2\zeta_3^2 \right\} \right]$$

$\mathcal{L}_S$ (symmetric deformation)

$$+ \left[ \frac{M}{2}\dot{\zeta}_4^2 + \frac{m}{2}\dot{\zeta}_5^2 - \frac{K}{2}\left\{ 2\zeta_4^2 - 2\zeta_4\zeta_5 + \zeta_5^2 \right\} \right]$$

$\mathcal{L}_A$ (antisymmetric deformation)

We could deal with each part separately.

Define $y_{1,3} = \sqrt{m}\,\zeta_{1,3}$ $\quad y_2 = \sqrt{m}\,\zeta_2$

$y_4 = \sqrt{M}\,\zeta_4$ $\quad y_5 = \sqrt{m}\,\zeta_5$

$$\mathcal{L}_S = \frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) - \frac{K}{2m}\left\{ y_1^2 + \frac{2m}{M}y_2^2 + 2y_3^2 - 2\sqrt{\frac{m}{M}}\left(y_1 + \sqrt{2}\,y_3\right)y_2 \right\}$$

$$\mathcal{L}_A = \frac{1}{2}\left(\dot{y}_4^2 + \dot{y}_5^2\right) - \frac{K}{2m}\left(2\frac{m}{M}y_4^2 + \eta_5^2 - 2\sqrt{\frac{m}{M}}\,y_4 y_5\right)$$

To get the eigenfrequencies and the eigen normal modes

Symmetric sector:

diagonalize $\begin{pmatrix} 1 & -\sqrt{\frac{m}{M}} & 0 \\ -\sqrt{\frac{m}{M}} & \frac{2m}{M} & -\sqrt{\frac{2m}{M}} \\ 0 & -\sqrt{\frac{2m}{M}} & 2 \end{pmatrix}$

~~It is going to have one zero eigenvalue~~

~~the corresponding eigenvector is~~ $\begin{pmatrix} \sqrt{\frac{m}{M}} \\ 1 \\ \sqrt{\frac{2m}{2M}} \end{pmatrix}$

$$\begin{vmatrix} (1-\lambda) & -\sqrt{\frac{m}{M}} & 0 \\ -\sqrt{\frac{m}{M}} & \left(\frac{2m}{M}-\lambda\right) & -\sqrt{\frac{2m}{M}} \\ 0 & -\sqrt{\frac{2m}{M}} & (2-\lambda) \end{vmatrix} = (1-\lambda)\left\{\left(\frac{2m}{M}-\lambda\right)(2-\lambda) - \frac{2m}{M}\right\} - \sqrt{\frac{m}{M}}\left(-\sqrt{\frac{m}{M}}\right)(2-\lambda)$$

$$= (\lambda-1)\left[\lambda^2 - 2\left(1+\frac{m}{M}\right)\lambda + \frac{2m}{M}\right] - \frac{m}{M}(\lambda-2)$$

$$= \lambda^3 - \lambda^2 - 2\left(1+\frac{m}{M}\right)\lambda^2 + 2\left(1+\frac{m}{M}\right)\lambda + \frac{2m}{M}\lambda - \cancel{\frac{2m}{M}} - \frac{m}{M}\lambda + \cancel{\frac{2m}{M}}$$

$$= \lambda\left[\lambda^2 - \left(3+\frac{2m}{M}\right)\lambda + \left(2+\frac{3m}{M}\right)\right]$$

The eigenvalues are $\lambda = 0,\ \dfrac{3+\frac{2m}{M} \pm \sqrt{\left(3+\frac{2m}{M}\right)^2 - 4\left(2+\frac{3m}{M}\right)}}{2}$

$= 0,\ \left(\frac{3}{2}+\frac{m}{M}\right) \pm \frac{1}{2}\sqrt{1+\frac{4m^2}{M^2}}$

The eigenfrequencies are given by $\omega^2 = \lambda \frac{k}{m}$

So $\omega^2 = 0,\ \frac{k}{m}\left[\frac{3}{2}+\frac{m}{M} \pm \frac{1}{2}\sqrt{1+\frac{4m^2}{M^2}}\right]$

One can find the other eigenvectors, It may be less work to use the frequencies and get the ~~eigen~~ normal modes ~~vectors~~ directly just using the symmetries

In the antisymmetric sector:

diagonalize

$$\begin{pmatrix} \frac{2m}{M} & \sqrt{\frac{m}{M}} \\ -\sqrt{\frac{m}{M}} & 1 \end{pmatrix}$$

$$\begin{vmatrix} \frac{2m}{M}-\lambda & -\sqrt{\frac{m}{M}} \\ \sqrt{\frac{m}{M}} & 1-\lambda \end{vmatrix} = (\lambda-1)\left(\lambda-\frac{2m}{M}\right)-\frac{m}{M}$$

$$= \lambda^2-\left(1+\frac{2m}{M}\right)\lambda+\frac{m}{M}$$

$$\lambda = \frac{1+\frac{2m}{M} \pm \sqrt{\left(1+\frac{2m}{M}\right)^2 - 4\frac{m}{M}}}{2},$$ corresponding eigenfrequencies are $\omega^2 = \frac{\lambda k}{m}$

$$\omega^2 = \frac{k}{m}\left[\frac{1}{2}+\frac{m}{M} \pm \frac{1}{2}\sqrt{1+\frac{4m^2}{M^2}}\right]$$

To get the normal modes we go back to

$$\underline{V}\,\underline{a} = \omega^2 \underline{T}\,\underline{a}$$

$$\underline{V} = \begin{pmatrix} k & -k & & & \\ -k & k & -k & & \\ & -k & k & -k & \\ & & -k & k & -k \\ & & & -k & k \end{pmatrix} \qquad T = \begin{pmatrix} m & & & & \\ & M & & & \\ & & m & & \\ & & & M & \\ & & & & m \end{pmatrix}$$

For the symetric sector ω's try

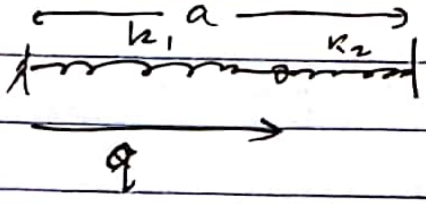
$$\underline{a} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \beta \\ \alpha \end{pmatrix} \Rightarrow \text{3 by 3 problem}$$

For the antisymmetric sector ~~to~~ ω's try

$$\underline{a} = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ -\beta \\ -\alpha \end{pmatrix} \Rightarrow \text{2 by 2 problems}$$

The expressions are messy but can be worked out.

Ch. 8 Prob 26




a) $\mathcal{L} = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k_1 q^2 - \frac{1}{2} k_2 (q-a)^2 \Rightarrow p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$

$$H = \frac{p^2}{2m} + \frac{1}{2} k_1 q^2 + \frac{1}{2} k_2 (q-a)^2$$

Since there is no explicit time dependence energy / Hamiltonian is conserved.

b) $q = Q + b\sin\omega t$

$$\mathcal{L} = \frac{1}{2} m (\dot{Q} + b\omega\cos\omega t)^2 - \frac{1}{2} k_1 (Q + b\sin\omega t)^2 - \frac{1}{2} k_2 (Q + b\sin\omega t - a)^2$$

$$P = m(\dot{Q} + b\omega\cos\omega t) = m\dot{q} = p$$

$$H_2 = P\dot{Q} - \mathcal{L} = \frac{1}{m} P (P - mb\omega\cos\omega t) - \frac{P^2}{2m} + \frac{1}{2} k_1 (Q + b\sin\omega t)^2 + \frac{1}{2} k_2 (Q + b\sin\omega t - a)^2$$

$$= \frac{P^2}{2m} + \frac{1}{2} k_1 (Q + b\sin\omega t)^2 + \frac{1}{2} k_2 (Q + b\sin\omega t - a)^2 - Pb\omega\cos\omega t$$

$$= \frac{p^2}{2m} + \frac{1}{2} k_1 q^2 + \frac{1}{2} k_2 (q-a)^2 - Pb\omega\cos\omega t$$

$$= H - Pb\omega\cos\omega t$$

This additional term spoils conservation of $H_2$.

Ch 9. Prob 6.

$$Q = \log(1+q^{1/2}\cos p)$$

$$P = 2(1+q^{1/2}\cos p)\, q^{1/2}\sin p$$

a) ~~One way to check canonical nature would be to see that~~

$$\boxed{[Q,P]} = 1$$

Calculate

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \frac{1}{1+q^{1/2}\cos p}\,\frac{1}{2}q^{-1/2}\cos p\,\{2\,q^{1/2}(\sin p)\,q^{1/2}\sin p + 2(1+q^{1/2}\cos p)\,q^{1/2}\cos p\}$$

$$- \frac{1}{1+q^{1/2}\cos p}\,q^{1/2}(-\sin p)\,\{2\cdot\frac{1}{2}q^{-1/2}\cos p\, q^{1/2}\sin p + 2(1+q^{1/2}\cos p)\frac{1}{2}q^{-1/2}\sin p\}$$

$$= \frac{1}{1+q^{1/2}\cos p}\Big[\cos p\,\{-q^{1/2}\sin^2 p + (1+q^{1/2}\cos p)\cos p\} + \sin p\,\{q^{1/2}\cos p\sin p + (1+q^{1/2}\cos p)\sin p\}\Big]$$

$$= \frac{1}{1+q^{1/2}\cos p}\left[(1+q^{1/2}\cos p)(\cos^2 p+\sin^2 p)\right] = 1$$

[QED]

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b) $$\frac{\partial F_3}{\partial p} = -q \qquad \frac{\partial F_3}{\partial Q} = -P$$

$$F_3 = -(e^Q - 1)^2 \tan p$$

$$q = -\frac{\partial F_3}{\partial p} = (e^Q - 1)^2 \sec^2 p$$

$$\Rightarrow \quad q \cos^2 p = (e^Q - 1)^2$$

$$\Rightarrow \quad e^Q = 1 + q^{1/2} \cos p$$

$$\Rightarrow \quad \boxed{Q = \log(1 + q^{1/2} \cos p)}$$

$$P = -\frac{\partial F_3}{\partial Q} = 2e^Q(e^Q - 1) \tan p$$

$$= 2(1 + q^{1/2} \cos p)\, q^{1/2} \cos p \tan p$$

$$= 2(1 + q^{1/2} \cos p)\, q^{1/2} \sin p$$

$\boxed{\text{QED}}$