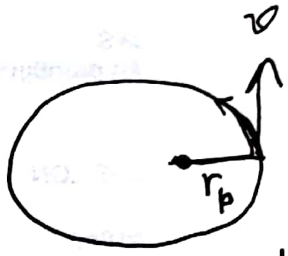
Problem Set 2

Answers

1. $$\frac{1}{r} = \frac{mk}{\ell^2}(1 + e\cos\theta)$$

$$\frac{1}{r_p} = \frac{mk}{\ell^2}(1+e)$$



So $\frac{\ell^2}{1+e}$ is fixed (same r_p), between the two orbits

$\ell = m v r_p$. Hence $\frac{v^2}{1+e}$ is fixed (same m, r_p).

$$\frac{v_f^2}{1+0} = \frac{v_i^2}{1+e}$$

circle

$$\Rightarrow \quad \frac{v_f}{v_i} = \sqrt{\frac{1}{1+e}} = \frac{1}{\sqrt{1+e}}$$

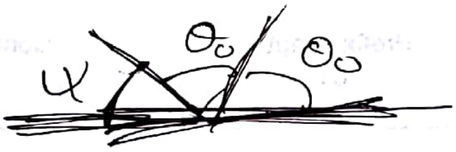
Scanned by CamScanner

2. a) $E = \frac{1}{2} m u^2 \qquad l = m u s$

b) $\frac{1}{r} = \frac{mk}{l^2}(e\cos\theta - 1)$

As $r \to \infty \qquad \theta \to \pm\theta_0$

$0 = e\cos\theta_0 - 1 \Rightarrow \cos\theta_0 = \frac{1}{e}$

c) 

$\psi = \pi - 2\theta_0$

or $\theta_0 = \frac{\pi}{2} - \frac{\psi}{2}$

d) $\cos\theta_0 = \frac{1}{e}$

$\Rightarrow \sin\frac{\psi}{2} = \frac{1}{e}$

e) $e^2 = 1 + \frac{2El^2}{mk^2} = 1 + \frac{2\frac{1}{2}mu^2 m^2u^2s^2}{mk^2}$

$= 1 + \left(\frac{mu^2s}{k}\right)^2$

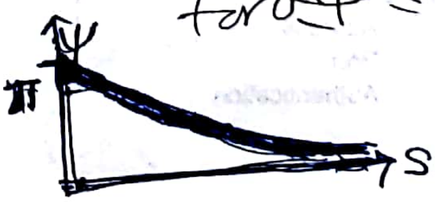
f) $\frac{1}{\sin^2\frac{\psi}{2}} = 1 + \left(\frac{mu^2s}{k}\right)^2$

$\left(\frac{mu^2s}{k}\right)^2 = \frac{1}{\sin^2\frac{\psi}{2}} - 1 = \frac{\cos^2\frac{\psi}{2}}{\sin^2\frac{\psi}{2}} = \left(\cot\frac{\psi}{2}\right)^2$

for $0 \le \psi \le \pi$

$\Rightarrow \tan\frac{\psi}{2} = \frac{k}{mu^2s}$

So $\psi = 2\tan^{-1}\left(\frac{k}{mu^2s}\right)$



Scanned by CamScanner

3. $A(\psi,\pi,\phi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \cos(\phi-\psi) & \sin(\phi-\psi) & 0 \\ \sin(\phi-\psi) & -\cos(\phi-\psi) & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $\alpha = \phi - \psi$

a) To find the axis, we need the eigenvector $\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$ with eigenvalue 1. Try

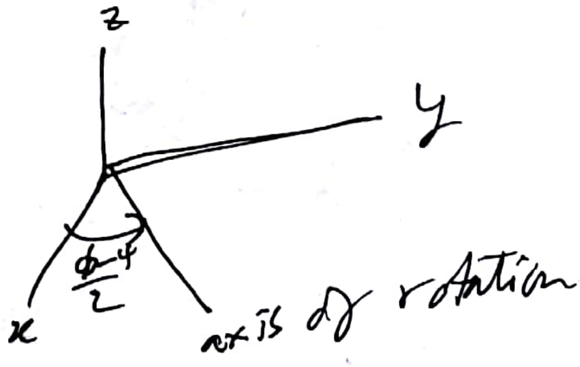
$A\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \Rightarrow$
$$\cos\alpha R_1 + \sin\alpha R_2 = R_1$$
$$\sin\alpha R_1 - \cos\alpha R_2 = R_2$$
$$R_3 = -R_3$$

Clearly $R_3 = 0$

$\cos\alpha R_1 + \sin\alpha R_2 = R_1 \Rightarrow (\cos\alpha - 1) R_1 + \sin\alpha R_2 = 0$

$$\Rightarrow \frac{R_1}{R_2} = \frac{\sin\alpha}{1-\cos\alpha} = \frac{2\sin\alpha/2 \cos\alpha/2}{2\sin^2\alpha/2} = \frac{\cos\alpha/2}{\sin\alpha/2}$$

So $\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha/2 \\ \sin\alpha/2 \\ 0 \end{pmatrix}$ is the normalized eigenvector with eigenvalue $=1$



The axis is in the x, y plane at an angle $\alpha/2 = (\phi-\psi)/2$ to x axis.

b) In the angle of rotation is Φ,

$$1+2\cos\Phi = \operatorname{Tr} A = \cos\alpha - \cos\alpha - 1 = -1$$

$$\Rightarrow \cos\Phi = -1 \Rightarrow \Phi = \pi$$

4. For matrices $(B+C)^n = B^n + (B^{n-1}C + BCB^{n-2} + \dots$
$+ CCB^{n-2} + CBCB^{n-3} + \dots$

If B, C do not commute we cannot collect it like binomial theorem for number.

If B, C commute

$$(B+C)^n = B^n + \binom{n}{1} B^{n-1}C + \binom{n}{2} B^{n-2}C^2 + \dots$$

$$= \sum_{r=0}^{\infty} \binom{n}{r} B^{n-r} C^r$$

Now $$e^B e^C = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{B^s}{s!} \frac{C^r}{r!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{r,s\geq 0 \\ r+s=n}} \frac{n!}{r!\,s!} B^s C^r$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} B^{n-r} C^r = \sum_{n=0}^{\infty} \frac{(B+C)^n}{n!}$$

$$= e^{B+C}$$

e^B transpose $$\widetilde{e^B} = \widetilde{\sum_{n=0}^{\infty} \frac{B^n}{n!}} = \sum_{n=0}^{\infty} \frac{(\tilde{B})^n}{n!} = e^{\tilde{B}}$$

If B is antisymmetric, $\tilde{A} = \widetilde{e^B} = e^{\tilde{B}} = e^{-B}$

$\tilde{A}A = e^{\tilde{B}} e^B = e^{-B} e^B = e^{-B+B} = e^0 = I$, when we use B and $-B$ commute.

Scanned by CamScanner