Course: Classical Mechanics
Prob Set 1
Solutions

1. Since $T$ and $\frac{\partial T}{\partial \dot{q}_j}$ are functions of $q_j$'s, $\dot{q}_j$'s and $t$, on them

$$\frac{d}{dt} = \sum_j \dot{q}_j \frac{\partial}{\partial q_j} + \sum_j \ddot{q}_j \frac{\partial}{\partial \dot{q}_j} + \frac{\partial}{\partial t}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{d}{dt} T = \overbrace{\left[\frac{\partial}{\partial \dot{q}_j}, \frac{d}{dt}\right]}^{\text{Commutator}} T + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}$$

$$= \left[\frac{\partial}{\partial \dot{q}_j}, \frac{d}{dt}\right] T + \frac{\partial T}{\partial q_j} + Q_j$$

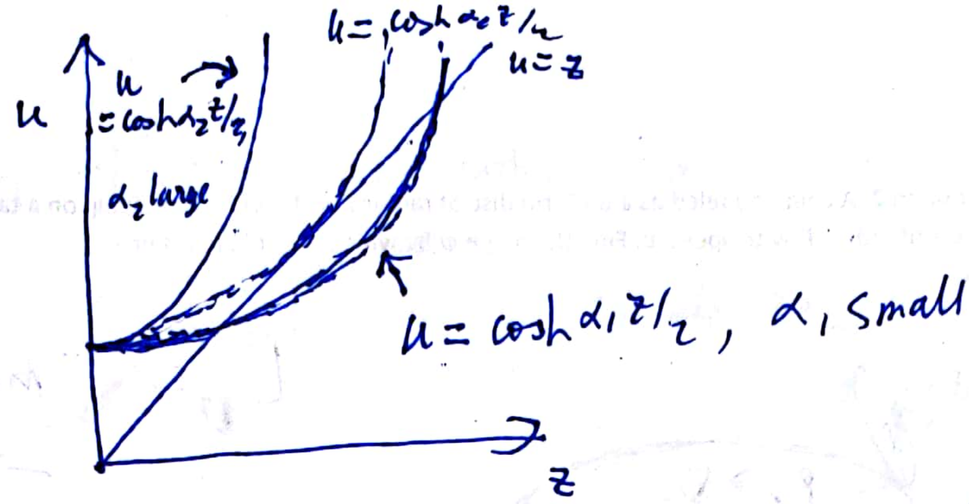
Since $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$

$$\left[\frac{\partial}{\partial \dot{q}_j}, \frac{d}{dt}\right] = \left[\frac{\partial}{\partial \dot{q}_j}, \sum_k \dot{q}_k \frac{\partial}{\partial q_k} + \sum_k \ddot{q}_k \frac{\partial}{\partial \dot{q}_k} + \frac{\partial}{\partial t}\right]$$

$$= \sum_k \left[\frac{\partial}{\partial \dot{q}_j}, \dot{q}_k\right] \frac{\partial}{\partial q_k} = \frac{\partial}{\partial q_j}$$

$$\Rightarrow \frac{\partial}{\partial \dot{q}_j}\dot{T} = 2\frac{\partial T}{\partial q_j} + Q_j \Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

2.a)




From the shape of the curves $u = \cosh \alpha z/2$ for different $\alpha$, one can see there would be two solutions for $\alpha < \alpha_c$ and no solution for $\alpha > \alpha_c$

b) $x = a\cosh\frac{y}{a}$ with $\frac{R}{a} = z$ and $L/R = \alpha$

$$\text{Area } A = 2\pi \int_{-L/2}^{L/2} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi a \int_{-\frac{L}{2}}^{\frac{L}{2}} \cosh\frac{y}{a}\sqrt{1+\sinh^2\frac{y}{a}}\, dy$$

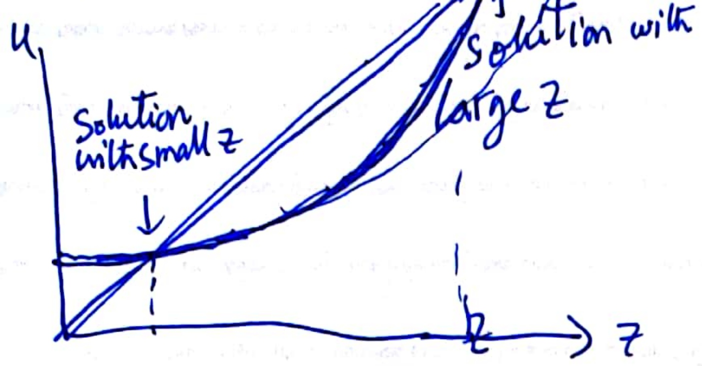
$$= 2\pi a \int_{-\frac{L}{2}}^{\frac{L}{2}} ch^2\frac{y}{a}\, dy = 2\pi a^2 \int_{-\frac{L}{2a}}^{\frac{L}{2a}} ch^2\theta\, d\theta = 2\pi a^2 \int_{-\alpha z/2}^{\alpha z/2} ch^2\theta\, d\theta$$

$$= 2\pi a^2 \int_{-\alpha z/2}^{\alpha z/2} \left(\frac{1+ch\,2\theta}{2}\right) d\theta = \pi a^2\left[\theta + \frac{sh\,2\theta}{2}\right]_{-\alpha z/2}^{\alpha z/2} = \pi a^2(\alpha z + sh\,\alpha z)$$

$$= \frac{\pi R^2}{z^2}\left[\alpha z + sh\,\alpha z\right]$$

This function is non monotonic so we need to look at the two solutions for a given $\alpha$.

Consider the case when $\alpha$ is small.




i) Small z solution:

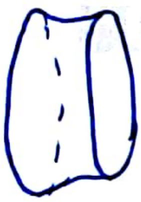
$$Z = \cosh \alpha Z/2 = 1 - \frac{\alpha^2 Z^2}{8} + \cdots$$

$$\Rightarrow Z = 1 - \frac{\alpha^2}{8} + O(\alpha^4)$$

$$A = \frac{\pi R^2}{\left(1-\frac{\alpha^2}{8}+\cdots\right)^2}\left[\alpha\left(1-\frac{\alpha^2}{8}+\cdots\right) + sh\,\alpha\left(1-\frac{\alpha^2}{8}+\cdots\right)\right] \simeq 2\pi R^2\alpha + O(\alpha^3)$$

$$\simeq \frac{2\pi R^2 L}{R} = 2\pi R L$$

$L/R \ll 1$



This soap surface is almost a cylinder

ii) The large Z solution

$$Z \simeq \cosh\left(\frac{\alpha Z}{2}\right) \simeq \frac{1}{2} e^{\alpha Z/2} \qquad \Rightarrow 2\ln(2Z) = \alpha Z$$

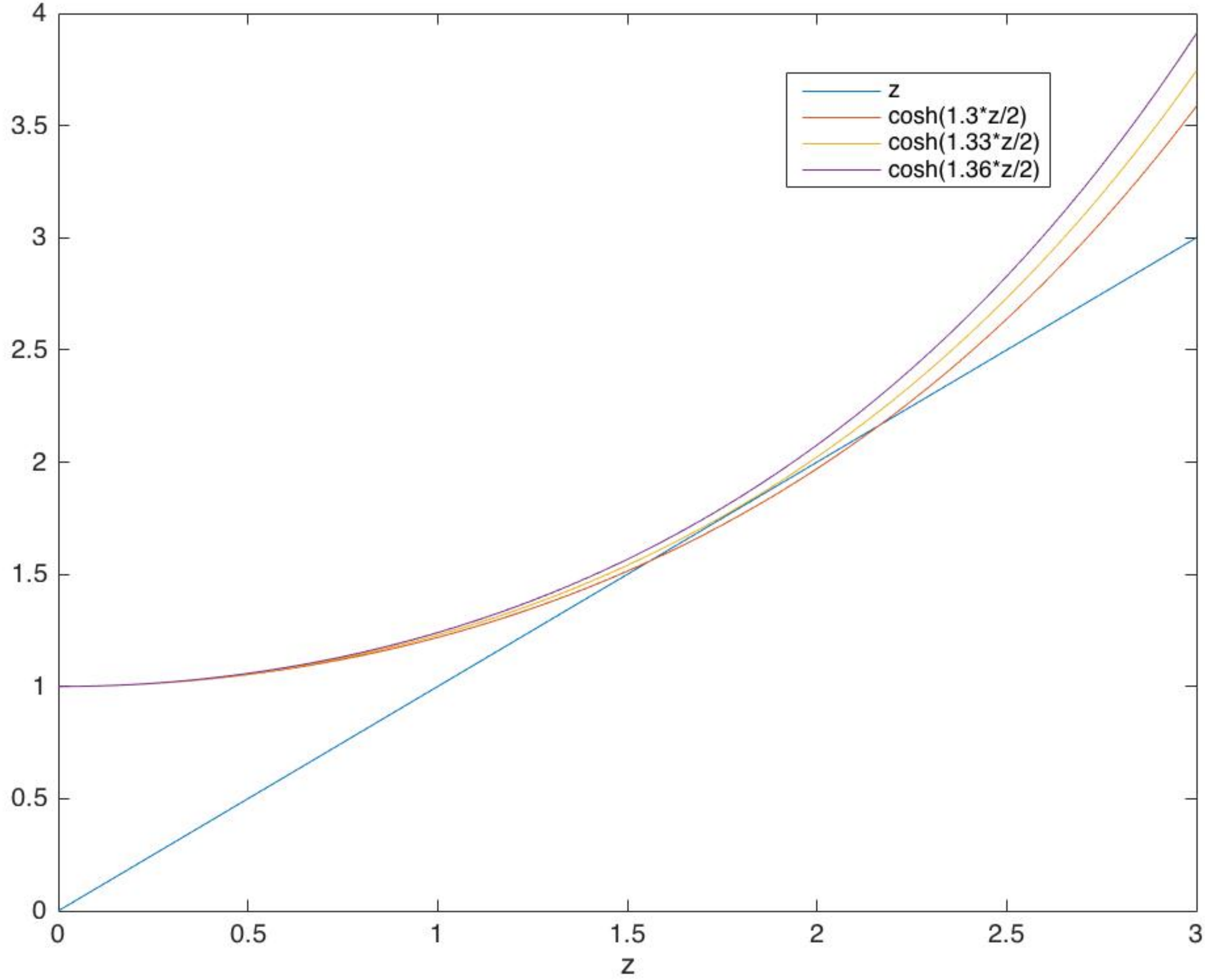
$$Z \simeq \frac{2}{\alpha}\frac{1}{\ln 2Z} \approx \frac{2/\alpha}{\ln(4/\alpha)}$$

$$A = \frac{\pi R^2}{Z^2}(\alpha Z + sh\,\alpha Z) \qquad sh\,\alpha Z \simeq \frac{e^{\alpha Z}}{2} \simeq \frac{1}{2}\left(e^{\alpha Z/2}\right)^2 \simeq \frac{1}{2}(2Z)^2 \simeq 2Z^2$$

$sh\,\alpha Z \gg \alpha Z \qquad A \simeq \frac{\pi R^2}{Z^2} \times 2Z^2 \simeq 2\pi R^2$

This is the narrow neck solution here, and the large Z solution is bigger area $2\pi R^2 \gg 2\pi R L$

c) $\alpha_c \approx 1.33$. See figure in the next page.



z
cosh(1.3*z/2)
cosh(1.33*z/2)
cosh(1.36*z/2)
0
0.5
1
1.5
2
2.5
3
3.5
4
z

3. a) $$\dot{\eta}(t) = \sqrt{\frac{2}{T_0}} \sum_n \frac{\pi n}{T_0} a_n \cos\frac{\pi n t}{T_0}$$

$$S = \int_0^{T_0} \left[ \frac{m}{2}\dot{\eta}^2 - \frac{m\omega^2}{2}\eta^2 \right] dt$$

$$= \frac{m}{2}\cdot\frac{2}{T_0}\Bigg[ \sum_{n,n'} \frac{\pi n}{T_0}\frac{\pi n'}{T_0} a_n a_{n'} \int_0^{T_0} \cos\frac{\pi n t}{T_0}\cos\frac{\pi n' t}{T_0}\, dt$$

$$- \sum_{n,n'} \omega^2 a_n a_{n'} \int_0^{T_0} \sin\frac{\pi n t}{T_0}\sin\frac{\pi n' t}{T_0}\, dt \Bigg]$$

$$\int_0^{T_0} \cos\frac{\pi n t}{T_0}\cos\frac{\pi n' t}{T_0}\, dt = \frac{1}{2}\int_0^{T_0}\left[\cos\frac{\pi(n+n')t}{T_0} + \cos\frac{\pi(n-n')t}{T_0}\right] dt$$

$$= \frac{1}{2} T_0 \left[\delta_{n+n',0} + \delta_{n-n',0}\right]$$

This is because $\int_0^{T_0} \cos\frac{\pi \nu t}{T_0}\, dt = \frac{T_0 \sin\frac{\pi\nu t}{T_0}}{\pi\nu}\Big|_0^{T_0} = 0$ if $\nu \neq 0$ and $\nu$ integer

$$\int_0^{T_0} \cos\frac{\pi \nu t}{T_0}\, dt = \int_0^{T_0} dt = T_0, \text{ if } \nu = 0$$