Rigid Body Eqn. of Motion

Consider either motion around a fixed point or motion around C.M.

Second case $T = \frac{1}{2} MV^2 +$ rotational K.E. around C.M.

[One could show that the angular velocity does not depend on the point of ref.]

$$\vec{L} = \sum_i m_i (\vec{r}_i \times \vec{v}_i) = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$\vec{v}_i = \vec{\omega} \times \vec{r}_i$

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

[bac-cab rule]

$$\vec{L} = \sum_i m_i \left[r_i^2 \vec{\omega} - \vec{r}_i (\vec{r}_i \cdot \omega) \right]$$

$$L_j = \sum_i m_i \left[r_i^2 \delta_{jk} - x_{ij} x_{ik} \right] \omega_k$$

Summ.n conv. on

$$= \underbrace{\int d^3\vec{r}\, \rho(\vec{r}) \left(r^2 \delta_{jk} - x_j x_k \right)}_{I_{jk}} \omega_k$$

← Inertia tensor

$$I_{xx} = \sum_i m_i \left(r_i^2 - x_i^2 \right)$$

$$I_{xy} = - \sum_i m_i x_i y_i$$

Symmetric matrix

$$\underline{\underline{L}} = \underline{\underline{I}}\ \underline{\omega}$$

Transformation properties

Summ.n conv.

$$x_i' = A_{ij} x_j \qquad \underline{x}' = \underline{A}\,\underline{x}$$

$I'_{jk} = a_{j\ell}$ [redacted] $a_{km} I_{\ell m}$

$$\underline{I}' = \underline{A}\,\underline{I}\,\tilde{\underline{A}}$$

[redacted] Moment of Inertia

Consider K. E.

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \frac{1}{2}\sum_i m_i v_i \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2}\sum_i m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i)$$

$$= \vec{\omega} \cdot \sum_i m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \frac{1}{2}\vec{\omega} \cdot \vec{L}$$

$$= \frac{1}{2}\, \underline{\omega} \cdot \underline{I} \cdot \underline{\omega}$$

In summation convention

$$T = \frac{1}{2}\, \omega_j I_{jk} \omega_k$$

Moment of inertia around an axis specified by $\hat{n}$.

$$\underline{\hat{n}} \cdot \underline{I} \cdot \underline{\hat{n}} = I(\hat{n})$$

$$T = \frac{1}{2} I(\hat{n})\,\omega^2$$

Corresponding radius of gyration $\frac{I(\hat{n})}{M} = R_0^2$

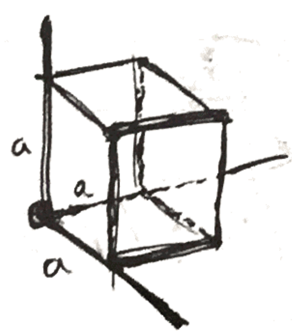
Examples:

$\langle x^2\rangle = \langle y^2\rangle$, [redacted] $\langle z^2\rangle = 0$

$\langle r^2\rangle = 2\langle x^2\rangle = 2\langle y^2\rangle$

$I_0 = M\langle r^2\rangle$ $\quad \langle xy\rangle = 0$ etc. [redacted]

$$\begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 \\ 0 & \frac{1}{2}I_0 & 0 \\ 0 & 0 & I_0 \end{pmatrix}$$



Cube of side a from corner

[redacted]

$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \frac{a^2}{3}$

$\langle xy\rangle = \langle yx\rangle = \langle zx\rangle = $ [redacted] $\frac{a^2}{4}$

If [redacted] $Ma^2 = I_0$

$$\underline{\underline{I}} = \begin{pmatrix} \frac{2}{3}I_0 & -\frac{1}{4}I_0 & -\frac{1}{4}I_0 \\ -\frac{1}{4}I_0 & \frac{2}{3}I_0 & -\frac{1}{4}I_0 \\ -\frac{1}{4} & -\frac{1}{4}I_0 & \frac{2}{3}I_0 \end{pmatrix}$$

Eigenvalues [redacted] and Principal axis transformation

Diagonalize I

$$I_D = R\, I\, \tilde{R}$$

$$I_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$I_i \geq 0$$

$$I_i = \int d^3r\, \rho(\vec{r})\, r^2_{\perp \text{ to } i}$$

For example:

$$I_3 = \sum_i m_i \left(r_i^2 - z_i^2\right) = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

Inertial ellipsoid

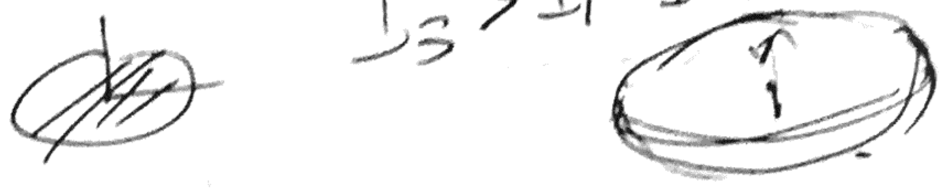
$$\underline{\rho} \cdot I \cdot \rho = 1$$

In normal form (Principal axis)

$$1 = I_1 \rho_1'^2 + I_2 \rho_2'^2 + I_3 \rho_3'^2$$

Rotates with the body.

$I_3 > I_1 = I_2$



Rigid body equation of mot

→ Euler equation

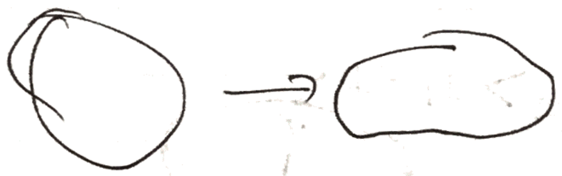
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_i \vec{r}_i \times \vec{p}_i = \sum_i \overset{0}{\dot{\vec{r}}_i \times \vec{p}_i} + \sum_i \vec{r}_i \times \vec{F}_i$$

$$= \vec{N}$$ ← Torque

In theory, that's it!
However

$$\underline{L} = \underset{\approx}{I} \cdot \omega$$

In space fixed coordinates
this depends of [redacted] rotation
itself.



$\underline{I}$ changes too!

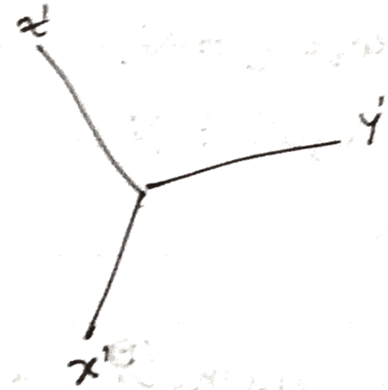
Alternative:

$$\left(\frac{dL}{dt}\right)_s = N$$

$$\left(\frac{dL}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \bar{\omega} \times \bar{L}$$

Now use the body fixed coordinates

[redacted]



$$\frac{dL'_i}{dt} + \epsilon_{ijk}\omega'_j L'_k = N'_i$$

If we use the principal axes as coordinates $\quad L'_i = I_i \omega'_i$ [no sum]

Drop primes:

$$I_1 \dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_1 = N_1$$

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

[redacted] $$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3$$

The Euler equations of motion.

In the special case, $I_1 = I_2$

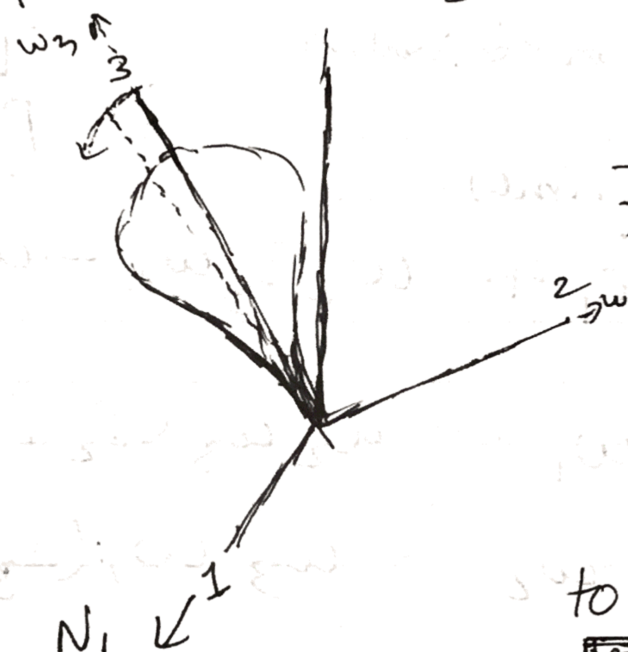
$$I_1 \dot{\omega}_1 + (I_3 - I_1)\,\omega_2\omega_3 = N_1$$

$$I_1 \dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 = N_2$$

$$I_3 \dot{\omega}_3 = N_3$$

For steady motion, namely $\dot{\omega}_i = 0$

We first have $N_3 = 0$.



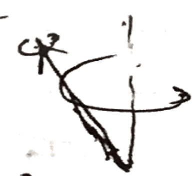
Also, since $I_1 = I_2$, we have freedom of rotation between axis 1 & 2.

Use that freedom to set $\omega_1 = 0$.

[redacted]

That implies $N_2 = 0$ and

$$N_1 = (I_3 - I_1)\,\omega_2\omega_3$$



[redacted]

We would return to this circumstance while discussing the heavy symmetric top

Torque free motion:

We could try solving $I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$ etc.

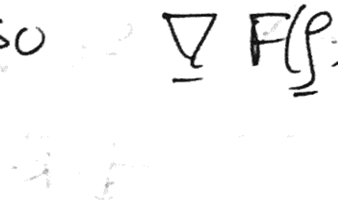
However, for qualitative understanding, it is better to use the following geometric construction

$$\frac{1}{2}\, \underline{\omega} \cdot \underline{I} \cdot \underline{\omega} = T.$$

Define $\underline{\rho} = \dfrac{\underline{\omega}}{\sqrt{2T}}$

Then $F(\underline{\rho}) = \underline{\rho} \cdot \underline{I} \cdot \underline{\rho} = 1$

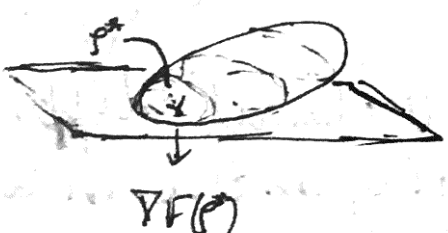
Using principal axis $F(\rho) = I_1 \rho_1'^2 + I_2 \rho_2'^2 + I_3 \rho_3'^2$



Inertia ellipsoid

Also $\underline{\nabla} F(\underline{\rho}) = 2\underline{I} \cdot \underline{\rho} = \sqrt{\dfrac{2}{T}}\, \underline{I} \cdot \underline{\omega}$

$= \sqrt{\dfrac{2}{T}}\, \underline{L}$



Tangent plane

Equation for the tangent plane

$$\underline{\nabla}_\rho F(\rho^*) \cdot \underline{\rho} = \underline{\nabla}_\rho F(\rho^*) \cdot \underline{\rho^*}$$

$$\sqrt{\frac{2}{T}}\, \underline{L} \cdot \rho = 2\left(\underline{I} \cdot \underline{\rho^*}\right) \cdot \rho^* = 2\, \underline{\rho^*} \cdot I \cdot \rho^*$$
$$= 2$$

Back to usual notation

$$\vec{\rho} \cdot \left(\frac{\vec{L}}{L}\right) = \frac{\sqrt{2T}}{L}$$

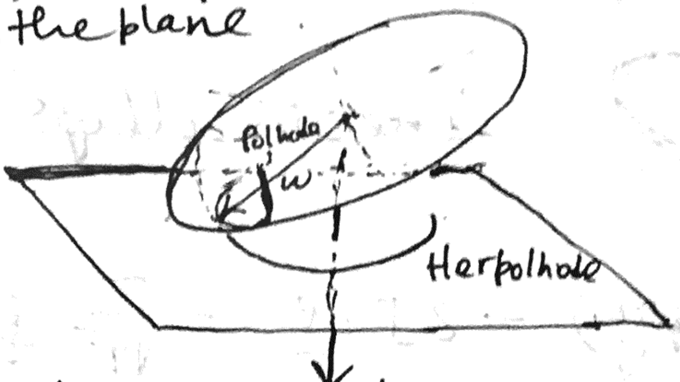
$\hat{L}$

Invariable Plane

$$\vec{\rho} \cdot \hat{L} = \frac{\sqrt{2T}}{L}$$

and $\underline{\rho} \cdot \underline{I} \cdot \underline{\rho} = 1$

The ellipsoid touches the plane



More over

$$\left(\frac{d\vec{\omega}}{dt}\right)_s = \left(\frac{d\vec{\omega}}{dt}\right)_b + \underbrace{\vec{\omega} \times \vec{\omega}}_{0}$$

So $\left|\left(\frac{d\vec{\omega}}{dt}\right)_s\right| = \left|\left(\frac{d\vec{\omega}}{dt}\right)_b\right|$

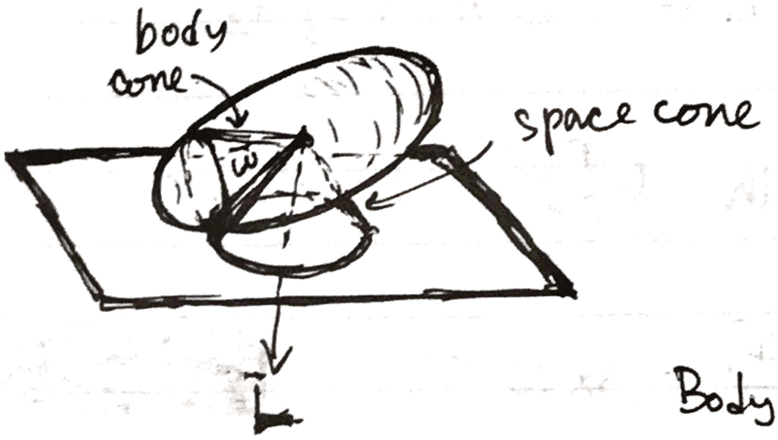
$\Rightarrow$ Rolling without slipping!

Poinsot Construction!

The polhode rolls without slipping on the herpolhode lying in the invariable plane

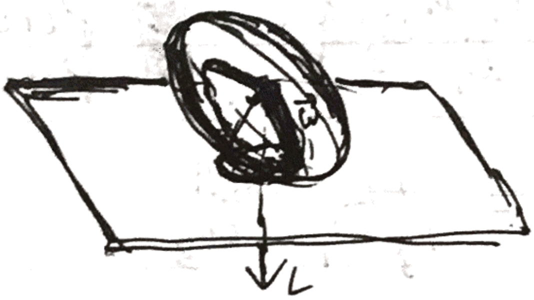
For $I_1 = I_2$

Inertia ellipsoid
= ellipsoid of revolution



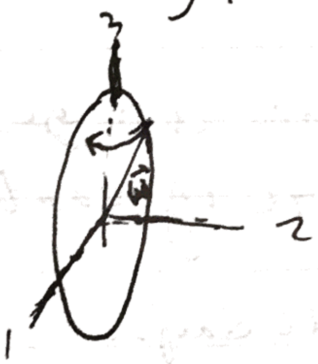
$I_3 < I_1$
Prolate

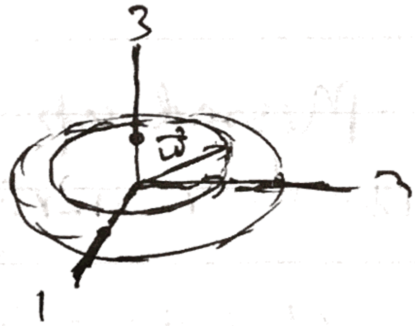
Body cone outside space cone



$I_3 > I_1$
Oblate
Body cone inside Space cone

Body fixed view





$$\left.\begin{aligned} I_1 \dot{\omega}_1 &= (I_1 - I_3)\,\omega_2 \omega_3 \\ I_1 \dot{\omega}_2 &= (I_3 - I_1)\,\omega_3 \omega_1 \\ I_3 \omega_3 &= 0 \end{aligned}\right\}$$

ω_3 constant. Define

$$\Omega = \frac{I_3 - I_1}{I_1}\,\omega_3$$

$$\dot{\omega}_1 = -\Omega \omega_2 \qquad \omega_1 = A\cos\Omega t$$

$$\dot{\omega}_2 = \Omega \omega_1 \qquad \omega_2 = A\sin\Omega t$$

$(\omega_1^2+\omega_2^2)$

$$T = \frac{1}{2} I_1 \underset{\uparrow}{A}^2 + \frac{1}{2} I_3 \omega_3^2$$

$$L^2 = I_1^2 A^2 + I_3^2 \omega_3^2$$

Earth  $I_3 > I_1$

$$\frac{I_3 - I_1}{I_1} = 3 \times 10^{-3}$$ [redacted]

$$\Omega = \frac{I_3 - I_1}{I_1} \omega_3 \Rightarrow$$ [redacted] Period of Precession

$$\frac{2\pi}{\Omega} = \frac{I_1}{I_3 - I_1} \frac{2\pi}{\omega_3}$$

$$= \frac{1}{3 \times 10^{-3}} \times 1\,\text{day}$$

$$\approx 300 \text{ days}$$

Messed up by seasonal changes and non-rigid nature of earth.

chandler wobble $\sim$ 420 days.

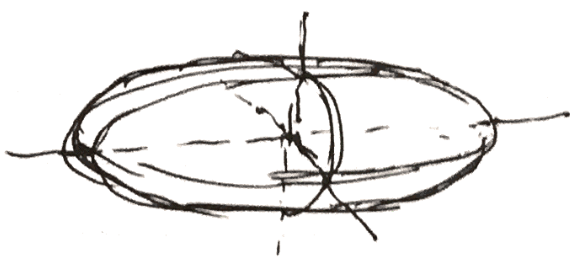
Notes on the other ellipsoid: Angular Momentum
→ Binet ellipsoid

$$\text{const} = T = \frac{L_x^2}{2I_1} + \frac{L_y^2}{2I_2} + \frac{L_z^2}{2I_3}$$

Let

$$I_3 \ll I_2 \ll I_1$$

$$\frac{L_x^2}{2TI_1} + \frac{L_y^2}{2TI_2} + \frac{L_z^2}{2TI_3} = 0$$



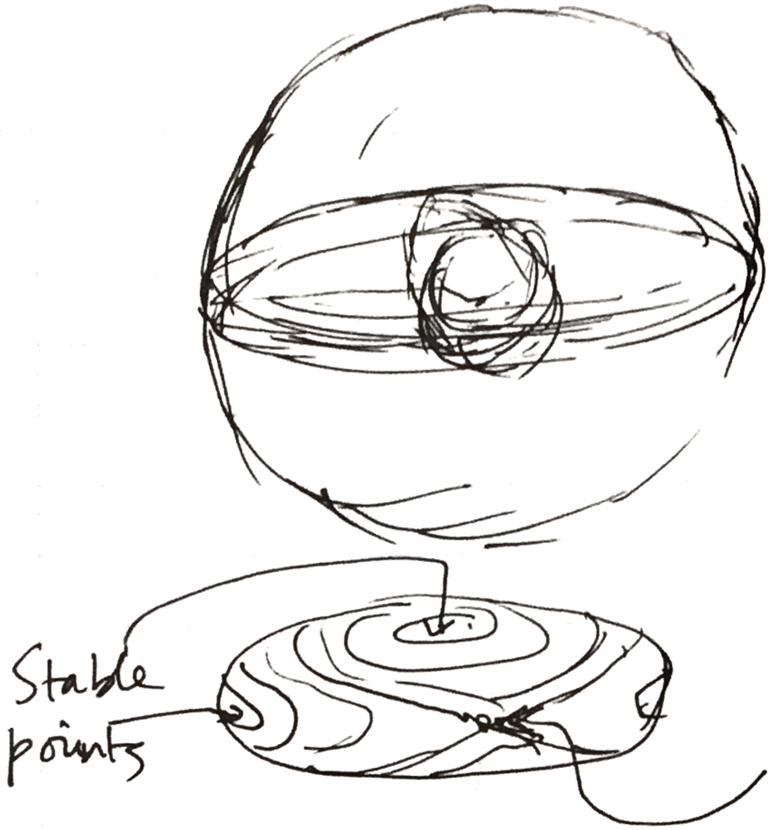
Angular momentum conservation

$$L_x^2 + L_y^2 + L_z^2 = L^2 = \text{const}$$

$$\frac{L_x^2}{L^2} + \frac{L_y^2}{L^2} + \frac{L_z^2}{L^2} = 1 \quad \rightarrow \text{Sphere}$$

Possible trajectories of $\vec{L}$ in the body fixed system.

⇒ intersection of the sphere with the ellipsoid



$\sqrt{2TI_3} < L < \sqrt{2TI_1}$

Perturbation of orbits $L_x = \pm\sqrt{2TI_1}$
$L_z = L_y = 0$

or $L_z = \pm\sqrt{2TI_1}$
$L_x, L_y = 0$

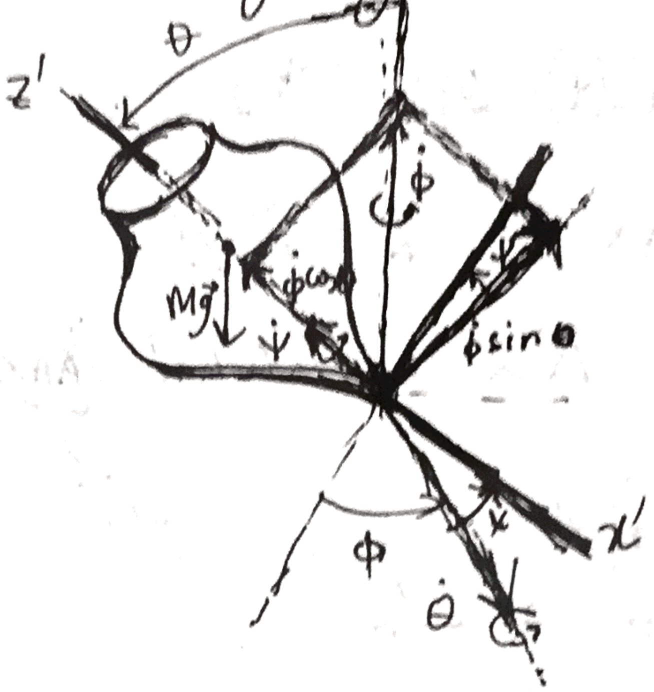
~~Sma~~ make small orbits

Perturbation of $L_y = \pm\sqrt{2TI_2}$
$L_x = L_z = 0$

Makes large orbits.

The ~~the~~ orbits almost reach the antipode
→ body flipping.

The heavy symmetrical top



$\dot\psi$ = rotation around z'

$\dot\phi$ = precession around vertical axis

$\dot\theta$ = nutation of the z' axis w.r.t. vertical axis.

Cheaty way

$$\omega_1'^2 + \omega_2'^2 = \dot\theta^2 + \dot\phi^2 \sin^2\theta$$

$$\omega_3'^2 = (\dot\psi + \dot\phi\cos\theta)^2$$

$$T = \tfrac{1}{2} I_1 (\omega_1'^2 + \omega_2'^2) + \tfrac{1}{2} I_3 \omega_3'^2$$

$$= \tfrac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2$$

More careful approach

$$A = B(\psi)C(\theta)D(\phi)$$

$$x' = A x \qquad x = \tilde{A} x'$$

$$\dot{x} = \dot{\tilde{A}} x' = \dot{\tilde{A}} A \tilde{A} x' = \underbrace{\dot{\tilde{A}} A}_{\Omega} x$$

$$A \dot{x} = \underbrace{A \dot{\tilde{A}}}_{\Omega'} x' \qquad \Omega' \rightarrow \omega' x$$

Potential energy

$$V = Mgl\cos\theta$$

$$\mathcal{L} = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta$$

Note that only coordinate that appears explicitly is θ. The coordinates ϕ and ψ are cyclic.

Conserved quantities

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3 = I_1 a$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta$$

$$= I_1 b$$

$$E = T + V = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_3}{2}\omega_3^2$$

$$+ Mgl\cos\theta$$

Task: Solve for $\dot\phi, \dot\psi$ in terms of p_ϕ, p_ψ, θ. Replace $\dot\phi$ in E in terms of that expression

$$\frac{I_3}{I_1}(\dot\psi + \dot\phi\cos\theta) = a$$

$$\frac{I_3}{I_1}\dot\psi\cos\theta + \left(\sin^2\theta + \frac{I_3}{I_1}\cos^2\theta\right)\dot\phi = b$$

$$\dot\phi\sin^2\theta = b - a\cos\theta$$

So $\dot\phi = \dfrac{b - a\cos\theta}{\sin^2\theta}$ $\quad \dot\psi = \dfrac{I_1 a}{I_3} - \dot\phi\cos\theta$

$$= \frac{I_1 a}{I_3} - \frac{\cos\theta(b - a\cos\theta)}{\sin^2\theta}$$

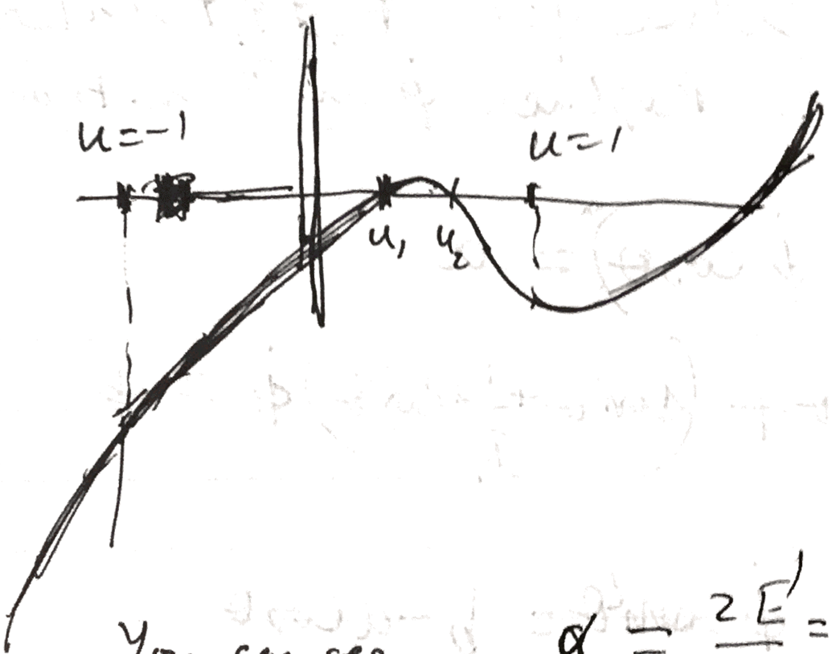
$$E' = E - \frac{I_3 \omega_3^2}{2}$$

$$= \frac{I_1 \dot\theta^2}{2} + \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2 + Mgl\cos\theta$$

$u = \cos\theta$ $\qquad \dot u = \sin\theta\,\dot\theta = \sqrt{1-u^2}\,\dot\theta$

$$E' = \frac{I_1 \dot u^2}{2(1-u^2)} + \frac{I_1}{2}\frac{(b-au)^2}{(1-u^2)} + Mgl\,u$$

$$\dot u^2 = (1-u^2)(\alpha - \beta u) - (b-au)^2$$
$$= f(u)$$



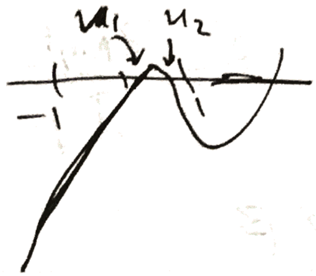
$f(u) \le 0$
at $u = \pm 1$

$f(u) \sim \beta u^3$
for large u

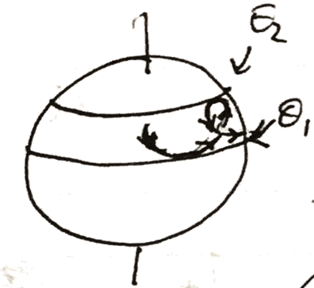
You can see $\quad \alpha = \frac{2E'}{I_1} = \frac{2E - I_3\omega_3^2}{I_1}$

and $\quad \beta = \frac{2Mgl}{I_1}$

$a = \frac{p_\psi}{I_1}, \quad b = \frac{p_\phi}{I_1}$



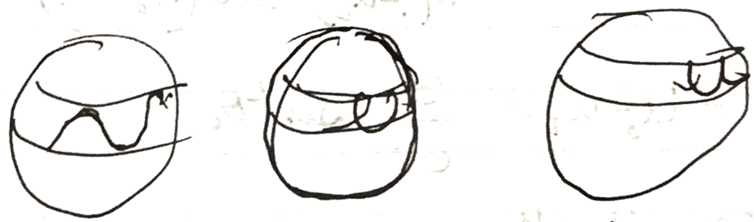
$u_1 = \cos\theta_1$
$u_2 = \cos\theta_2$



nutation

Turning /Bounding angles.

Possible motion



"Dropping a Spinning top"

Starts with $\dot{\theta} = \dot{\phi} = 0$

Call initial $\theta = \theta_0$ $\quad u = u_0$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \qquad b - a\cos\theta = 0 \qquad b = a u_0$$

Also $\dot{\theta}^2 = 0$ $\qquad Mgl\cos\theta = E'$

$$\alpha = \beta u_0$$

$$f(u) = (1-u^2)\,\beta(u_0 - u) - a^2(u_0-u)^2$$
$$= (u_0 - u)\left[\beta(1-u^2) - a^2(u_0 - u)\right]$$

$$a = \frac{I_3 \omega_3}{I_1} \qquad \beta = \frac{2Mgl}{I_1}$$

If [redacted] $a^2 >> \beta$

$f(u_1) = 0$

$$\beta(1-u_0^2) = a^2(u_0 - u_1)$$

$$x_1 = u_0 - u_1 = \frac{\beta}{a^2} \sin^2\theta_0$$

$$x = u_0 - u$$

$$\dot{x}^2 = a^2 x(x_1 - x) = a^2\left[\left(\frac{x_1}{2}\right)^2 - \left(\frac{x_1}{2} - x\right)^2\right]$$

$$\dot{y}^2 = a^2\left[\frac{x_1^2}{4} - y^2\right] \overset{\frac{d}{dt}}{\Rightarrow} 2\dot{y}\ddot{y} = -2a^2 y\dot{y}$$

$$\Rightarrow \quad \ddot{y} = -a^2 y$$

[redacted] $y \sim \cos at$

x[redacted] $= \frac{x_1}{2}(1 - \cos at)$

$$a = \frac{I_3}{I_1}\omega_3$$

$$\dot{\phi} = \frac{a(u_0 - u)}{\sin^2\theta_0} = \frac{a\,x}{\sin^2\theta_0} = \frac{a x_1}{2\sin^2\theta_0}(1 - \cos at)$$

$$= \frac{\beta}{2a}(1 - \cos at)$$

$$\bar{\dot{\phi}} = \frac{\beta}{2a}$$