The Kinematics of Rigid Body Motion

Distance between any pair of particles fixed

$$\boxed{r_{ij} = c_{ij}}$$

3N variables, Superficially ${}^{N}C_{2}$ constraints

Note that if we gave distances from 3 reference points, a point gets fixed upto discrete choices. Thus, if I give coordinates of only 3 points in a rigid body, we are set.

Point 1 : $\boxed{\text{3 coordinates.}}$

Point 2 : Subject to $r_{12} = c_{12}$

on a sphere $\boxed{\text{2 more coordinates}}$

Point 3 : Subject to $r_{13} = c_{13}$

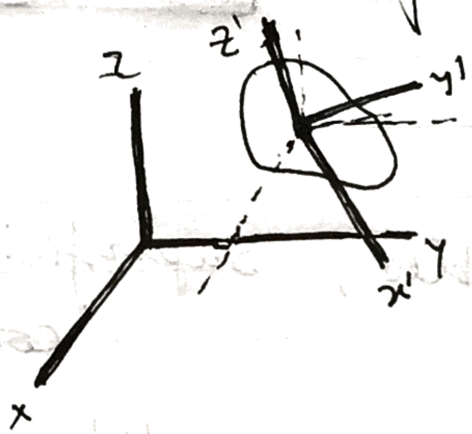
$r_{23} = c_{23}$

On a circle $\boxed{\text{1 more coordinate}}$

6 coordinates total!

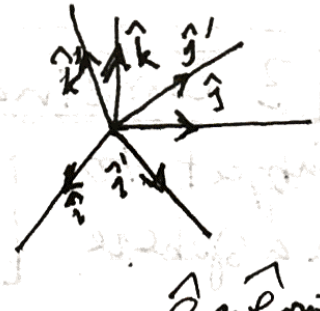
3 of these coordinates could be thought of as specifying the center of mass.

3 others are for orientation.



Body-fixed axes (primed)

How do we specify the orientation of the new axis system?



Call $\hat{i} = \hat{e}_1$

$\hat{j} = \hat{e}_2$

$\hat{k} = \hat{e}_3$

$\hat{i}' = \hat{e}_1'$ etc

$\hat{e}_\ell \cdot \hat{e}_m = \delta_{\ell m}$, $\hat{e}'_\ell \cdot \hat{e}'_m = \delta_{\ell m}$

Define

$$\cos\theta_{\ell m} = \hat{e}'_\ell \cdot \hat{e}_m$$

$\theta_{\ell m}$ is the angle between these basis vectors.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

$$x' = \vec{r}\cdot\hat{i}' = \hat{i}'\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \cos\theta_{11}\, x + \cos\theta_{12}\, y + \cos\theta_{13}\, z$$

It is true for any vector $\vec{G}$

$$G_{x'} = \vec{G}\cdot\hat{i}' = \cos\theta_{11} G_x + \cos\theta_{12} G_y + \cos\theta_{13} G_z$$

etc...

Of course $\{\theta_{lm}\}$ 9 in number not 3.

They are not independent!

$$\hat{e}_m = \sum_{l} \cos\theta_{lm}\, \hat{e}_l'$$

$$\delta_{mm'} = \hat{e}_m\cdot\hat{e}_{m'} = \sum_{l l'} \cos\theta_{lm} \cos\theta_{l'm'}\, \hat{e}_l\cdot\hat{e}_{l'}$$

$$= \sum_{\ell \ell'} \cos\theta_{\ell m} \cos\theta_{\ell' m'} \delta_{\ell \ell'}$$

$$= \sum_{\ell} \cos\theta_{\ell m} \cos\theta_{\ell m'}$$

So $\sum_{\ell} \cos\theta_{\ell m} \cos\theta_{\ell m'} = \delta_{m m'}$

$m = m' \Rightarrow$ 3 constraints for $m = 1, 2, 3$

$m \neq m' \Rightarrow$ 3 constraints because ~~no~~ there are 3 pairs

$9 - 3 - 3 = 3$ independent directional cosines.

Need other coordinates for Lagrangian description.

The orthogonality conditions has to be understood for ~~as~~ their usefulness in general.

Orthogonal Transformations

Notⁿ $a_{ij} = \cos\theta_{ij}$

$x'_i = a_{ij}\, x_j$ [Summation convention!]

$(\equiv \sum_j a_{ij} x_j)$

Means

$x'_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$

$x'_2 = \ldots$ etc.

$\sum_i x_i^2$ represented as $x_i x_i$.

Note that, under rigid rotations,

$x'_i x'_i = x_i x_i$

Or $a_{ij} a_{ik} x_j x_k = x_i x_i$

for any values of $\{x_i\}$.

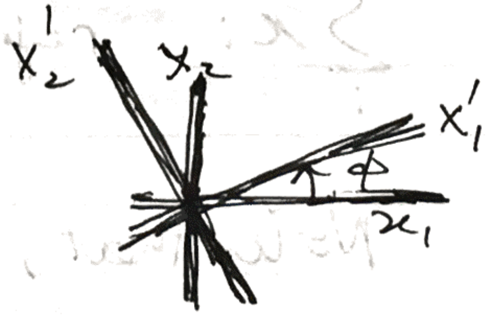
$\Rightarrow \quad a_{ij} a_{ik} = \delta_{jk}$

We could also use matrix notn

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

'Orthogonality' condition: columns are orthonormal vectors

Example: Rotation around $z (= x_3)$ axis



[redacted]

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$a_{11} = \cos\phi$ $\qquad a_{12} = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$

$a_{21} = \cos\left(\frac{\pi}{2} + \phi\right)$
$= -\sin\phi$ $\qquad a_{22} = \cos\phi$

$$A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

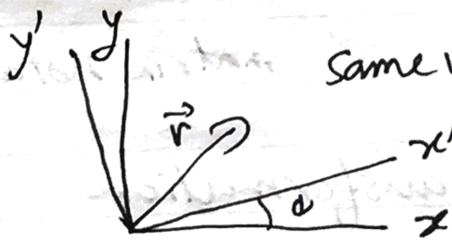
The column orthonormality condition satisfied

because $\cos^2\phi + \sin^2\phi = 1$

and $\cos\phi \sin\phi - \sin\phi\cos\phi = 0$

Active and passive view

Passive view : Coord change

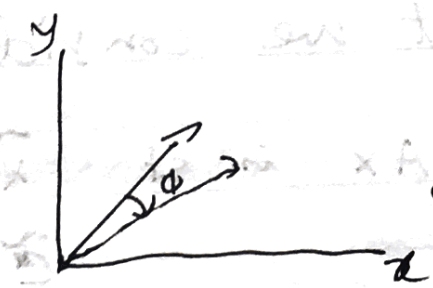


same vector

Active
[redacted] view

[redacted]

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$$



same coordinate system

Back to matrices

Combining transformations

$x'_k = b_{kj} x_j \qquad x''_i = a_{ik} x'_k$

$x''_i = c_{ij} x_j = a_{ik} b_{kj} x_j$

$c_{ij} = a_{ik} b_{kj}$

$C = A\,B$ matrix notation

Inverse transformation

$x' = A x \qquad A^{-1} x' = x$

Note that we consider transformations where

$x' = A x$ is s.t. $\tilde{x}' x' = \tilde{x} x$

$\tilde{\ }$ for transpose

$\tilde{x}' x' = \tilde{x} \tilde{A} A x$. If that is $\tilde{x} x$ for every x, then $\tilde{A} A = \mathbb{1}$, or $\tilde{A} = A^{-1}$

Orthogonal matrices

Few more definitions $A_{ij} = A_{ji}$ symmetric

$A_{ij} = -A_{ij}$ antisymmetric/ skew symmetric

If $y = Cx$ and $y' = Ay$
$x' = Ax$
in a different coordinate system,
then

$$y' = Ay = ACx = ACA^{-1}Ax$$

$$= ACA^{-1}x'$$

So in the new coordinate system

$C \to C' = ACA^{-1}$ Similarity transformation

Determinants $|A|$

$$|AB| = |A|\,|B|$$

$|I| = 1$ so $|A^{-1}|\,|A| = 1 \Rightarrow |A^{-1}| = 1/|A|$

Note that for any matrix $|A| = |\tilde{A}|$

For orthogonal matrice $\tilde{A}A = I$

$\Rightarrow 1 = |\tilde{A}|\,|A| = |A|^2$. So $|A| = \pm 1$

$|A| = +1 \rightarrow$ Continuously connected to I
(Rotn) $\rightarrow SO(3)$

$|A| = -1 \rightarrow$ Inversion & Rotn

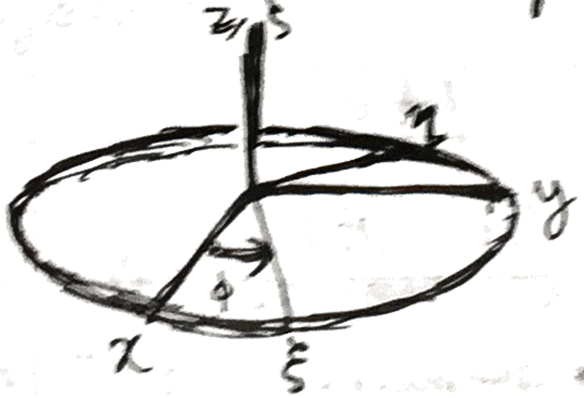
Similarity transformation does not change the value of determinant

[redacted]

$$|C'| = |ACA^{-1}|$$
$$= |A|\,|C|\,|A^{-1}|$$
$$= \cancel{|A|}\,|C|\left(\frac{1}{\cancel{|A|}}\right)$$
$$= |C|$$

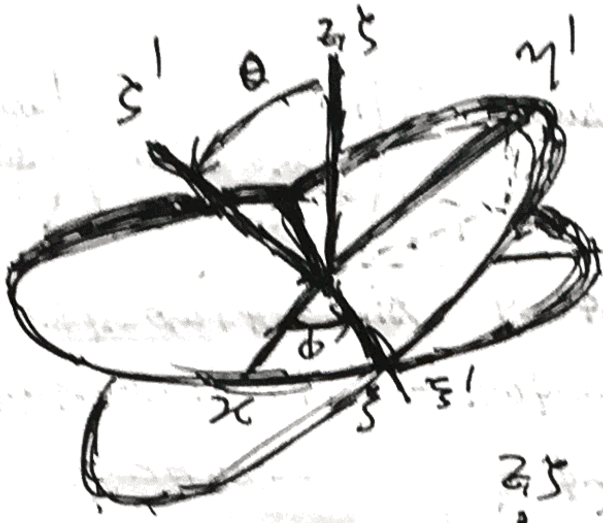
Describing Rigid Rotn

Euler Angles



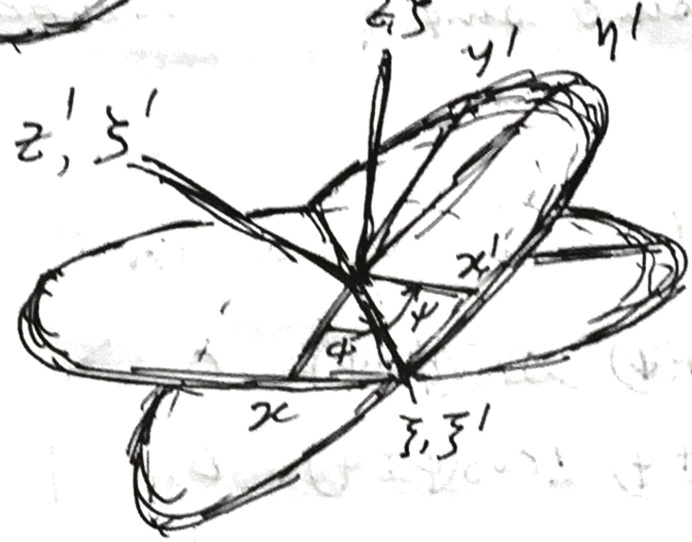
$$\underline{\xi} = D\underline{x}$$

$$\underline{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$



$$\underline{\xi}' = C\,\underline{\xi}$$

$$\underline{\xi}' = \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix}$$



$$\underline{x}' = B\,\underline{\xi}'$$

$$\underline{x}' = A\,\underline{x} \Rightarrow A = BCD$$

$$D = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1 \end{pmatrix}$$

$$A = BCD = \begin{pmatrix} \cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\cos\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Note that when $\theta = 0$ or $\theta = \pi$

$$\begin{pmatrix} \cos(\phi \pm \psi) & \sin(\phi \pm \psi) & 0 \\ \mp\sin(\phi \pm \psi) & \pm\cos(\phi \pm \psi) & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

Two dim $\rightarrow$ 1 dim (Singular point in coordinate system)

A^{-1} is just $\tilde{A}$.

We will discuss other ways [redacted] of parametriz[redacted] rotations, like Cayley Klein [redacted] parameters later.

Euler's Theorem: Any displacement of a rigid body with one point fixed is a rotation about some axis.

To show that we will look at the structure of eigenvalues and eigenvectors of a 3×3 orthogonal matrix with determinant 1.

Before we go there let us examine the structure of eigenvalues, eigenvectors a matrix that represents a particular rotation: rotation around z axis by angle ϕ ($\phi \neq 0$).

$$A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} \cos\phi - \lambda & \sin\phi & 0 \\ -\sin\phi & \cos\phi - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(\cos\phi - \lambda)^2 + \sin\phi^2\right]$$

$$= -(\lambda - 1)(\lambda^2 - 2\cos\phi\,\lambda + 1)$$

$$= -(\lambda - 1)(\lambda - e^{i\phi})(\lambda - e^{-i\phi})$$

So the eigen values are $1, e^{i\phi}, e^{-i\phi}$.
What are the eigenvectors?

$\lambda = 1$ is easy

$(A - \lambda I)\underline{R} = 0$

$$\begin{pmatrix} \cos\phi - 1 & \sin\phi & 0 \\ -\sin\phi & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Esp'ly for $|\cos\phi| < 1$ $\quad \underline{R} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

In general, the axis should be an eigenvector for eigenvalue 1.

What about $\lambda = e^{i\phi} = \cos\phi + i\sin\phi$

$$\begin{pmatrix} -i\sin\phi & \sin\phi & 0 \\ -\sin\phi & -i\sin\phi & 0 \\ 0 & 0 & 1 - e^{i\phi} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Once more for $|\cos\phi| < 1$.

$z = 0 \qquad x + iy = 0$

$$\underline{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = e^{i\phi} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \xrightarrow{\text{complex conj}} A \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

So, for a non-trivial rotn, there is the eigenvector with eigenvalue 1. That eigenvector is .

Now back to Euler's Theorem

$\tilde{A} A = I \qquad |A| = 1$

Say $AR = \lambda R$

Taking complex conjugate $\quad \underset{\substack{\uparrow \\ \text{real}}}{A} R^* = \lambda^* R^*$

Also $\quad \tilde{R} \tilde{A} = \lambda \tilde{R}$

Multiplying $\quad \tilde{R} \tilde{A} A R^* = \lambda \lambda^* \tilde{R} R^*$

$\tilde{R} R^* = \lambda \lambda^* \tilde{R} R^*$

$\tilde{R} R^* = (x\,y\,z)\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = |x|^2 + |y|^2 + |z|^2 \neq 0$

So $\quad |\lambda|^2 = 1$

$0 = |A - \lambda I| = |A| + (\,)\lambda + (\,)\lambda^2 - \lambda^3$

If eigenvalues are $\lambda_1, \lambda_2, \lambda_3 \qquad |A| = \lambda_1 \lambda_2 \lambda_3$

Either all are real or one real and two others are complex conj:

If all are real and $|\lambda| = 1$
only possibilities are

$1, 1, 1 \rightarrow$ zero angle rotn around any axis

$1, -1, -1$

Rotn by π around the eigenvector for eigenvalue 1.

If complex $|\lambda| = 1 \Rightarrow \lambda = e^{i\Phi}$

$\Phi \neq 0, \pi$

$1, e^{i\Phi}, e^{-i\Phi}$

Rotation by angle Φ around the eigen vector for eigenvalue 1.

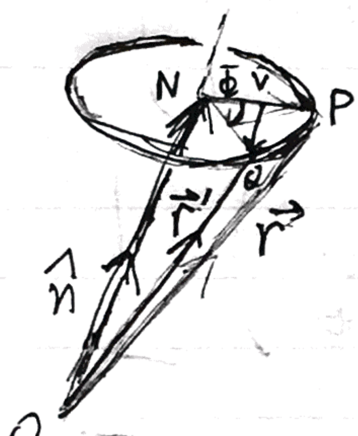
Note that $Tr(A) = a_{ii}$ is an invariant

$Tr A = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\Phi$

$\hookrightarrow 1 + e^{i\Phi} + e^{-i\Phi} \nearrow$

Chasles' Thm: The most general displacement of a rigid body is a translation plus a rotn.

Finite Rotn (active view)



$$\vec{r}\,' = \vec{ON} + \vec{NV} + \vec{VQ}$$

$$= \hat{n}(\hat{n}\cdot\vec{r}) + [\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})]\cos\Phi + (\vec{r}\times\hat{n})\sin\Phi$$

$$= \vec{r}\cos\Phi + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\Phi) + \vec{r}\times\hat{n}\sin\Phi$$

Relating Rotn Angle to Euler's

$$1 + 2\cos\Phi = \mathrm{Tr}\,A = \mathrm{Tr}\left[\underbrace{\begin{pmatrix}\cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1\end{pmatrix}}_{B}\underbrace{\begin{pmatrix}1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta\end{pmatrix}}_{C}\underbrace{\begin{pmatrix}\cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1\end{pmatrix}}_{D}\right]$$

Put them together

But $\mathrm{Tr}(BCD) = \mathrm{Tr}(DBC)$

$$B_{ij}\,C_{kl}\,D_{li} = D_{li}\,B_{ij}\,C_{jl}$$

$$\mathrm{Tr}\left[\begin{pmatrix} \cos(\phi+\psi) & \sin\phi+\psi & 0 \\ -\sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\right]$$

Calculate just the diagonal [redacted] on the product matrix

$$\begin{pmatrix} \cos(\phi+\psi) & - & - \\ - & \cos\theta\cos(\phi+\psi) & - \\ - & - & \cos\theta \end{pmatrix}$$

$1 + 2\cos\bar{\Phi}$ [redacted] $=$ [redacted] $\cos(\phi+\psi) + \cos\theta\cos(\phi+\psi) + \cos\theta$ [redacted]

$1+2\cos\bar{\Phi} =$ [redacted] $\{1+\cos(\phi+\psi)\}\{1+\cos\theta\} - 1$ [redacted]

[redacted] $2(1+\cos\bar{\Phi}) =$ [redacted] $\{1+\cos(\phi+\psi)\}(1+\cos\theta)$

$4\cos^2\frac{\bar{\Phi}}{2} =$ [redacted] $2\cos^2\frac{\phi+\psi}{2} \cdot 2\cos^2\frac{\theta}{2}$

$$\cos\frac{\Phi}{2} = \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}$$

Infinitesimal Rotations

A close to identity

$A = I + \epsilon$ ← small

$$x'_i = (\delta_{ij} + \epsilon_{ij})x_j = x_i + \epsilon_{ij}x_j$$

$$x' = (I+\epsilon)x$$

Product of two rotation

$$(I+\epsilon_1)(I+\epsilon_2) = 1 + \epsilon_1 + \epsilon_2 + \underbrace{\epsilon_1\epsilon_2}_{\text{small}}$$

↑ non commutative

$$\tilde{A}A = I$$

$$\Rightarrow (I+\tilde{\epsilon})(I+\epsilon) = I$$

$$\approx I + \tilde{\epsilon} + \epsilon$$

$$\tilde{\epsilon} = -\epsilon$$ Antisymmetric matrix

Remember rotation around z axis

$$\begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

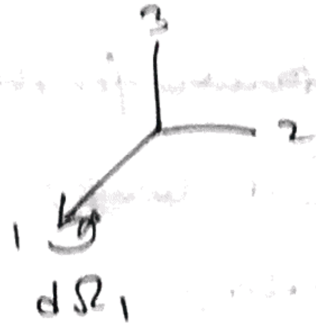
$$= \begin{pmatrix} 1-\phi^2/2+\cdots & \phi - \cdots & \\ -\phi+\cdots & 1-\phi^2/2+\cdots & \\ & & 1 \end{pmatrix}$$

Dropping second order onward.

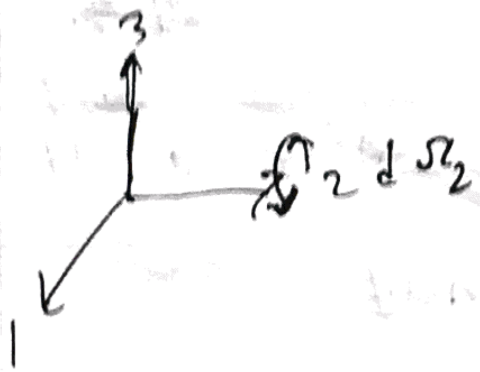
$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} & \phi & \\ -\phi & & \\ & & \end{pmatrix}}$$

Let us represent it as

$$\begin{pmatrix} 0 & d\Omega_3 & 0 \\ -d\Omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d\Omega_1 \\ 0 & -d\Omega_1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & -d\Omega_2 \\ 0 & \cancel{d\Omega_2} & 0 \\ d\Omega_2 & 0 & 0 \end{pmatrix}$$

Generally

$$\epsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$d\vec{r} = \vec{r}' - \vec{r} = \epsilon \vec{r}$$

$$dx_1 = x_2 \, d\Omega_3 - x_3 \, d\Omega_2$$
$$dx_2 = x_3 \, d\Omega_1 - x_1 \, d\Omega_3$$
$$dx_3 = x_1 \, d\Omega_2 - x_2 \, d\Omega_1$$

$$d\vec{r} = \vec{r} \times d\vec{\Omega}$$

However from the rotation formula

$$\vec{r}' = \vec{r}\cos\Phi + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\Phi) + \vec{r}\times\hat{n}\sin\Phi$$

Rotation by $d\Phi$ $\quad \cos\Phi \to 1 - \frac{d\Phi^2}{2} + \dots \approx 1$

$\sin\Phi \to d\Phi$

$$\vec{r}' = \vec{r} + \vec{r}\times\hat{n}\,d\Phi$$

So $d\vec{\Omega} = \hat{n}\,d\Phi$

In the matrix form

$\epsilon = d\Omega_k M_k$ matrices

$dr = \vec{r}\times d\vec{\Omega}$

$\Rightarrow \quad dx_i = \varepsilon_{ijk} x_j\, d\Omega_k = (\varepsilon_{kij}\, d\Omega_k)\, x_j$

$(M_k)_{ij} = \varepsilon_{kij}$

Note that $(M_1 M_2)_{ij} = \varepsilon_{1ik}\varepsilon_{2kj}$

$$= \varepsilon_{k1i}\varepsilon_{kj2}$$

$$= \delta_{1j}\delta_{i2} - \delta_{ij}\underbrace{\delta_{12}}_{0}$$

$$\left([M_1, M_2]\right)_{ij} = \delta_{1j}\delta_{i2} - \delta_{2j}\delta_{i1}$$

$$= \varepsilon_{k12}\varepsilon_{kij} = \varepsilon_{3ij} = (M_3)_{ij}$$

In fact

$$\left.\begin{aligned} [M_1, M_2] &= M_3 \\ [M_2, M_3] &= M_1 \\ [M_3, M_1] &= M_2 \end{aligned}\right\} \text{Lie algebra}$$

$$[M_i, M_j] = \varepsilon_{ijk} M_k$$

Lie Algebra.

A quick word about Cayley-Klein parameters.

Notice that $\frac{i\sigma_x}{2}$ $\frac{i\sigma_y}{2}$ $\frac{i\sigma_z}{2}$

Satisfy the same algebra.

$e^{i\vec{\sigma}\cdot\vec{\theta}/2}$

$\in SU(2)$

generators

$\frac{i\vec{\sigma}}{2}\cdot d\vec{\theta}$

spin $\frac{1}{2}$

$U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}$

Fundamental repr

If $U \in SU(2)$

$U\ \vec{\sigma}\cdot\vec{x}\ U^{+} = \vec{\sigma}\cdot\vec{x}'$

$U^{+} = U^{-1}$ $|U| = 1$

Adjoint representation — spin 1 representations

$\sigma_x = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$ $\sigma_y = \begin{pmatrix} & -i\\ i & \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}$

$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix} z & x-iy\\ x+iy & -z\end{pmatrix}\begin{pmatrix}\alpha^{*} & \gamma^{*}\\ \beta^{*} & \delta^{*}\end{pmatrix}$

But $\alpha\beta^* + \gamma\delta^* = 0$

$\alpha\gamma^* + \beta\delta^* = 0$

$|\alpha|^2 + |\gamma|^2 = 1$

$|\alpha|^2 + |\beta|^2 = 1$

$|\beta|^2 + |\delta|^2 = 1$

and $\alpha\delta - \beta\gamma = 1$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$

$\gamma = -\beta^*$ $\quad \delta = \alpha^*$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \beta\delta(x+iy) - \alpha\gamma(x-iy) + (\alpha\delta+\beta\gamma)z & -\beta^2(x+iy) + \alpha^2(x-iy) - 2\alpha\beta z \\ \delta^2(x+iy) - \gamma^2(x-iy) + 2\gamma\delta z & -\beta\delta(x+iy) + \alpha\gamma(x-iy) - (\alpha\delta+\beta\gamma)z \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e_0 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + ie_1 \begin{pmatrix} & 1 \\ i & \end{pmatrix} + ie_2 \begin{pmatrix} & -i \\ i & \end{pmatrix} + ie_3 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\alpha = e_0 + ie_3$ $\quad \beta = e_2 + ie_1$

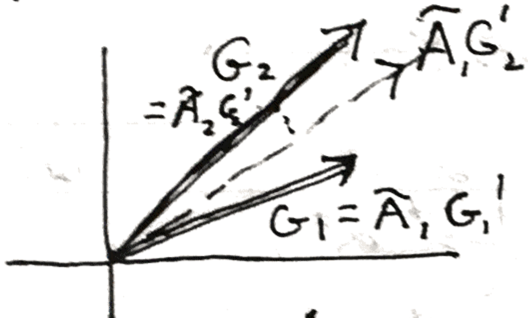
$|\alpha|^2 + |\beta|^2 = 1 \Rightarrow e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$

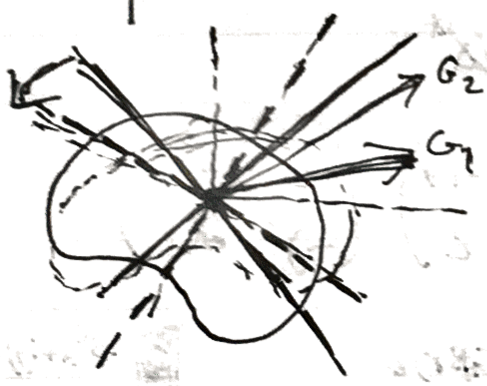
3-sphere

Useful in gyroscopic calculations!

Rate of change: Body fixed and space

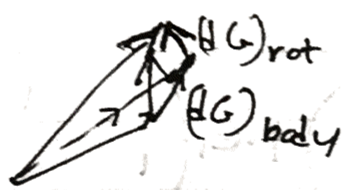
Space





Body fixed

$$(dG)_{space} = (dG)_{body} + (dG)_{rot}$$



More formally: $G(t) = \tilde{A}(t)\, G'(t)$

$$\frac{dG(t)}{dt} = \tilde{A}(t)\frac{dG'(t)}{dt} + \frac{d\tilde{A}(t)}{dt}\tilde{A}(t)\,\tilde{A}(t)G(t)$$

$$\left(\frac{dG}{dt}\right)_{\cancel{body}\ space} = \left(\frac{dG}{dt}\right)_{body} + \left(\frac{d\hat{A}}{dt}A\right)G$$

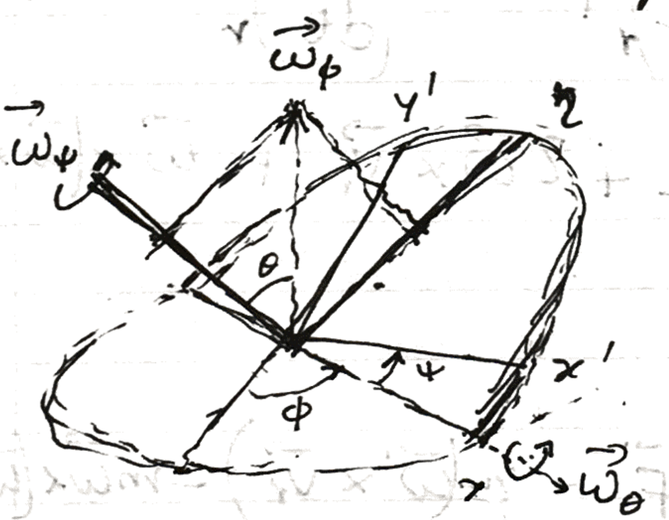
$$\tilde{A}(t+dt) = (I+\tilde{\epsilon})\tilde{A}(t)$$

$$\tilde{\epsilon} = -\epsilon$$ and $$-\epsilon_{ij} = \epsilon_{ikj}\, d\Omega_k$$

$$(d\vec{G})_{space} = (d\vec{G})_{body} + d\vec{\Omega} \times \vec{G}.$$ $$d\vec{\Omega} = \vec{\omega}\, dt$$

$$\left(\frac{d}{dt}\right)_S = \left(\frac{d}{dt}\right)_r + \omega \times$$

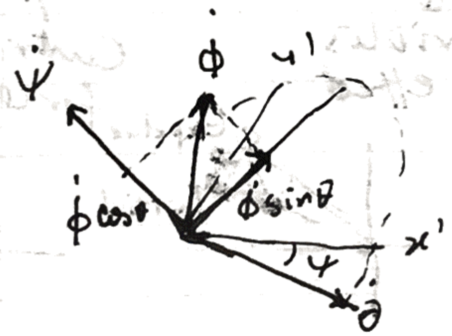
rotating system



$$\omega_{x'} = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_{y'} = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_{z'} = \dot\phi \cos\psi + \dot\psi$$



Rotating (Non inertial) Frame

$\vec{\omega}$ = constant

$$\left(\frac{d\vec{r}_s}{dt}\right)_s = \left(\frac{d\vec{r}_s}{dt}\right)_r + \vec{\omega}\times\vec{r}_r$$

$$\vec{V}_s = \vec{V}_p + \vec{\omega}\times\vec{r}_s$$

$$\vec{a}_s = \left(\frac{d\vec{V}_s}{dt}\right)_s = \left(\frac{d\vec{V}_s}{dt}\right)_r + \vec{\omega}\times\vec{V}_s$$

$$= \left(\frac{d^2\vec{V}_r}{dt^2}\right)_r + \vec{\omega}\times\left(\frac{dr_s}{dt}\right)_r + \vec{\omega}\times(\vec{V}_r + \vec{\omega}\times\vec{r}_s)$$

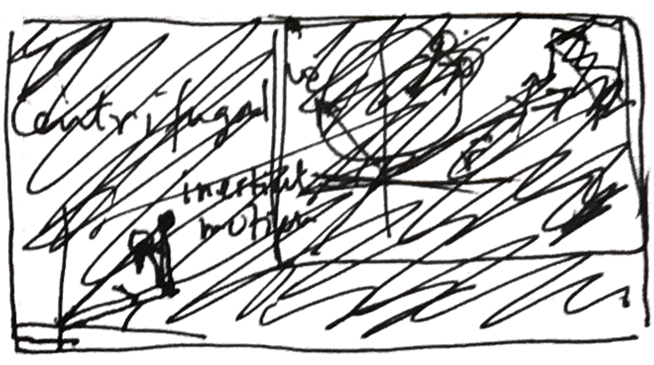
$$= \vec{a}_r + 2\vec{\omega}\times\vec{V}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r}_s)$$

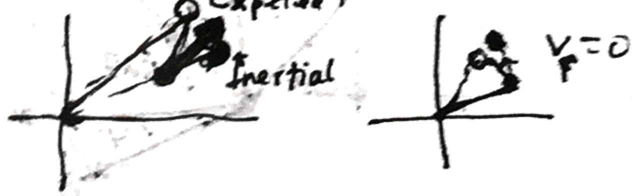
$$\vec{F} = m\vec{a}_s$$

$$m\vec{a}_r = \vec{F} - 2m(\vec{\omega}\times\vec{V}_r) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

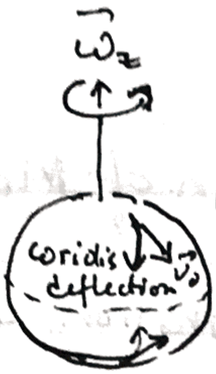
Coriolis effect

Centrifugal force



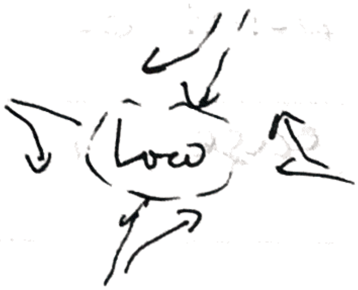


$\vec{F} = 0$



Northern hemisphere

North
Clockwise
rotn of cyclones



Foucault's Pendulum