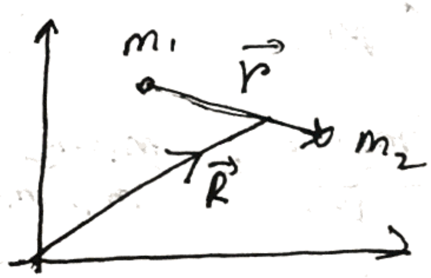
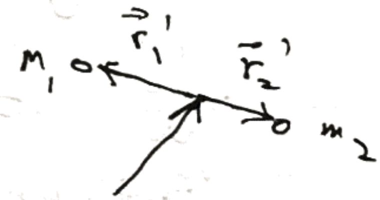
The Central Force Problem

[redacted]



$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{r}_i\,' = \vec{r}_i - \vec{R}$$



We saw that

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2} M \vec{V}^2 + \frac{1}{2}\sum_i m_i \vec{v}_i\,'^2$$

$$M = \sum_i m_i \qquad \vec{V} = \dot{\vec{R}} \qquad \vec{v}_i\,' = \dot{\vec{r}}_i\,'$$

$$\vec{r}_1\,' = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{(m_1+m_2)\vec{r}_1 - (m_1\vec{r}_1 + m_2\vec{r}_2)}{m_1+m_2}$$

[redacted] $$= \frac{m_2(\vec{r}_1 - \vec{r}_2)}{m_1+m_2} = -\frac{m_2}{m_1+m_2}\vec{r} = -\frac{m_2\vec{r}}{M}$$

$$\vec{r}_2\,' = \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1+m_2} = \frac{m_1}{m_1+m_2}\vec{r} = \frac{m_1\vec{r}}{M}$$

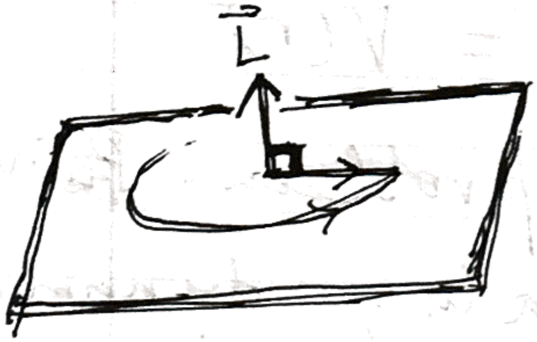
Conserved quantities and first integrals.

$$L = T - V = \frac{1}{2} m \dot{\vec{r}}^2 - V(r)$$

Symmetries: Rotational invariance
Time-translation "

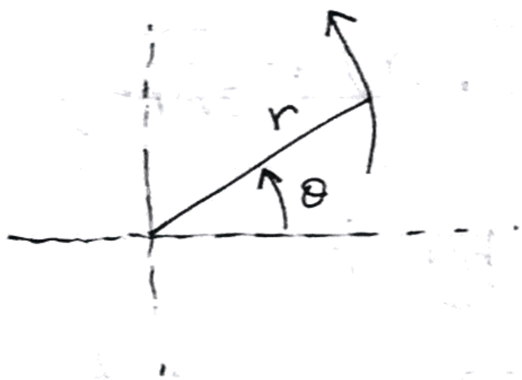
→ Angular momentum conservation
Energy "

$$\vec{L} = \vec{r} \times \vec{p} \text{ is conserved}$$

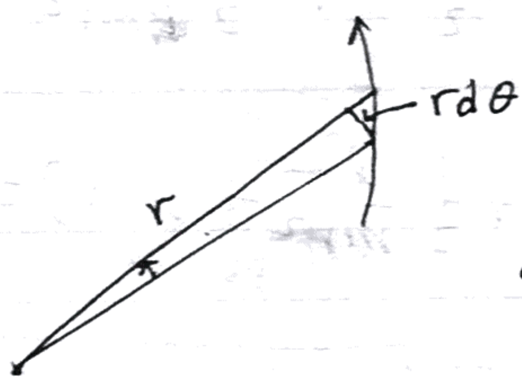


$\vec{r}$ is always $\perp$ to $\vec{L}$
so $\vec{r}$ lies in a plane.

Set up polar coordinates.



$$L = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2) - V(r)$$



$\vec r \times \vec p\, dt = m\vec r \times \vec v\, dt$
$= m\vec r \times d\vec r$

$$dA = \frac{1}{2} r^2 d\theta$$

$\frac{dA}{dt} = \frac{|\vec L|}{2m}$ [redacted] $=$ Const.

Anyway $p_\theta = m r^2 \dot\theta = l = |\vec L|$

Other equation

$$\frac{d}{dt}(m\dot r) = m r\dot\theta^2 - \frac{d}{dr}V(r)$$

$= \frac{l^2}{m r^3}$ [redacted] $\frac{d}{dr}V(r)$

using $\dot\theta = \frac{l}{mr^2}$

$$\frac{d}{dt} m\dot{r} = -\frac{d}{dr}\left(V(r) + \frac{\ell^2}{2mr^2}\right)$$

Multiply by $\dot{r}$

$$\frac{d}{dt}\left(\frac{m\dot{r}^2}{2}\right) = -\frac{d}{dt}\left(V(r) + \frac{\ell^2}{2mr^2}\right)$$

$$\frac{d}{dt}\left[\frac{m\dot{r}^2}{2} + \frac{1}{2}\frac{\ell^2}{\text{[redacted]}\, mr^2} + V(r)\right] = 0$$

$$\frac{m\dot{r}^2}{2} + \frac{1}{2}\frac{\ell^2}{mr^2} + V(r) = E \quad = \text{constant.}$$

$$\dot{r}\ \text{[redacted]} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}$$

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

$t = t(r)$

$\Rightarrow r = r(t)$

$$\dot{\theta} = \frac{l}{m r^2(t)}$$

$$\theta = l \int_0^t \frac{dt'}{m\, r^2(t')} + \theta_0$$

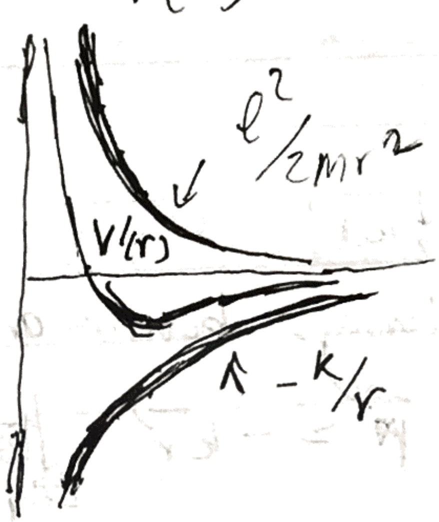
Equivalent one-dim problem

$$E = \frac{1}{2} m \dot{r}^2 + V'(r)$$

$$V'(r) = V(r) + \frac{l^2}{2 m r^2}$$

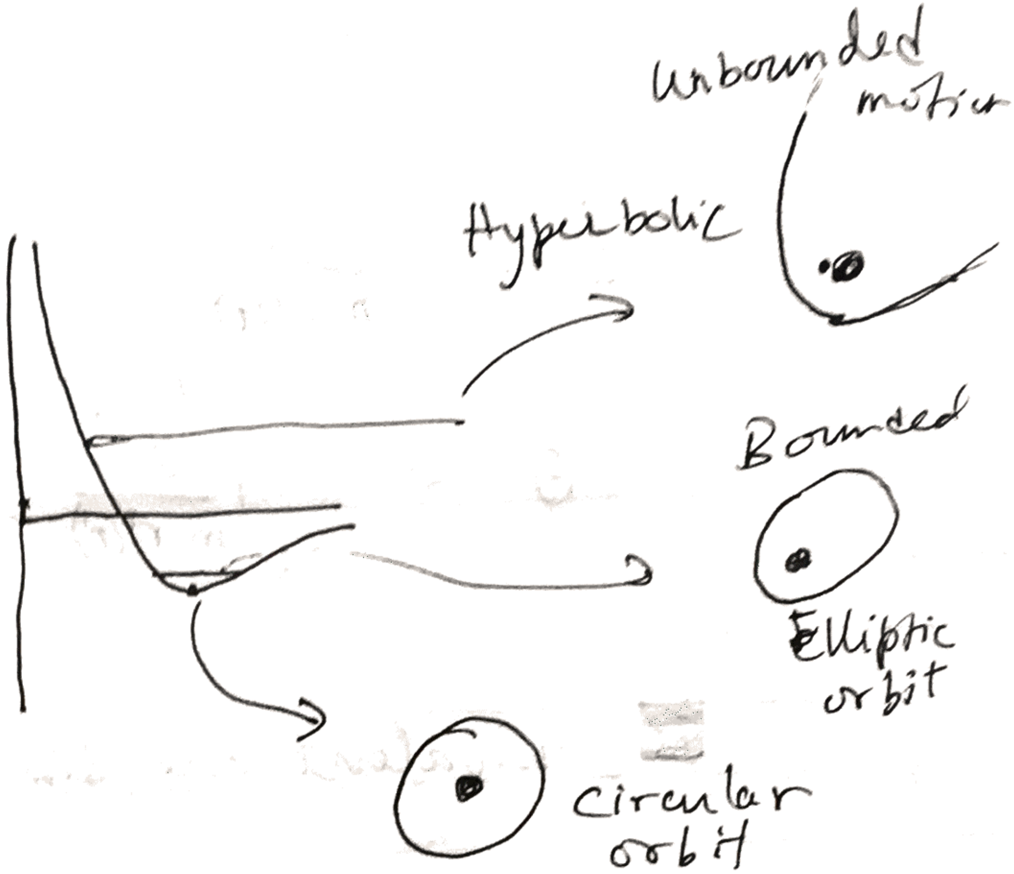
Consider the special case

$$V(r) = -\frac{k}{r}$$



Large r $\quad -k/r$ dominant

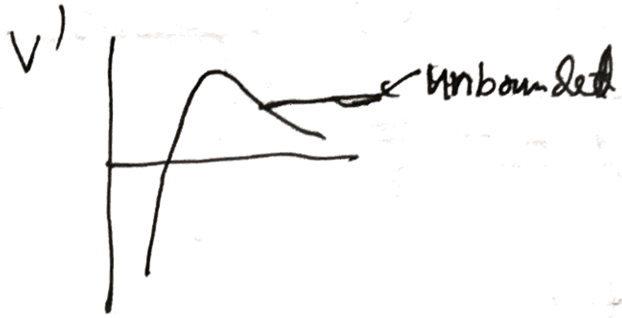
Small r $\quad \frac{l^2}{2mr^2}$ dominant



V(r) isn't slower than $1/r^2$ } stable large bounded orbits are hard to are hard to get.

Other potentials

Ugly: $V(r) = -\frac{a}{r^3}$



Nice! $V(r) = \frac{1}{2} k r^2$

always bound orbits

$\vec{p} = -k\vec{r}$

Virial Theorem:

More general than central motion, Gives information about statistical averages

Define $G = \sum_i \vec{p}_i \cdot \vec{r}_i$

$$\dot{G} = \sum_i \dot{\vec{r}}_i \cdot \vec{p}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i$$

$$= \sum_i m \vec{v}_i^{\,2} + \sum_i \vec{F}_i \cdot \vec{r}_i$$

$$= 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

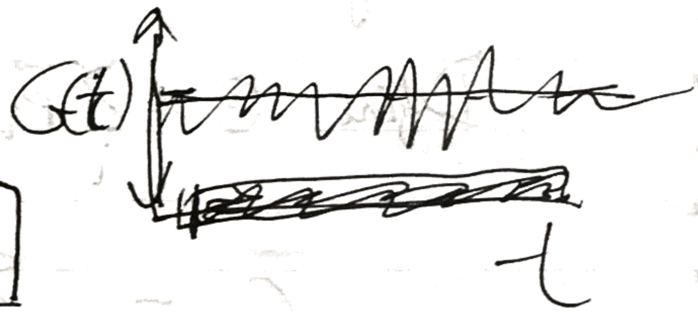
Integrating from $t=0$ $t=\tau$ and dividing by τ.

$$\frac{1}{\tau}\left[G(\tau) - G(0)\right] = 2\bar{T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

Where $\dfrac{\int_0^\tau \theta(t)\,dt}{\int_0^\tau dt}$ is the time average $\bar{\theta}$

If the motion is statistically periodic

$$\frac{G(\tau)-G(0)}{\tau} \to 0$$



as $\tau \to \infty$

$$\Rightarrow \boxed{\overline{T} = -\frac{1}{2}\overline{\sum \vec{F}_i \cdot \vec{r}_i}}$$

Applicn to ideal gas:

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

$$\frac{3}{2} N k_B T = \frac{1}{2}\int \vec{r}\cdot(P\hat{n}\,dA)$$

$$= \frac{1}{2} P \int \vec{r}\cdot d\vec{A}$$

$$= \frac{1}{2} P \int \vec{\nabla}\cdot\vec{r}\,dV$$

$$= \frac{3}{2} P V$$

$$\boxed{PV = Nk_BT}$$

In principle, with weak collisions one could 'correct' the equation of state for non-ideal gases

For conservative systems

$$F_i = \nabla_i V$$

$$\overline{T} = \frac{1}{2}\sum_i \vec{r}_i \cdot \vec{\nabla}_i V$$

Aside on homogeneous functions

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$$

Degree k homogeneous functions

Put $\lambda = 1+\varepsilon$ $\quad\quad$ ε small

$$f(x_1+\varepsilon x_1, x_2+\varepsilon x_2, \ldots, x_N+\varepsilon x_N)$$

$$= (1+\varepsilon)^k f(x_1, \ldots, x_N)$$

Expanding upto first order in ε we get

$$\sum_i x_i \frac{\partial}{\partial x_i} f = k f \text{, Euler's theorem}$$

So, if $V(\lambda \vec{r}_1, \ldots, \lambda \vec{r}_N) = \lambda^{n+1} V(\quad)$

[Force goes as λ^n]

$$\bar{T} = \overline{\frac{1}{2}\sum_i \vec{r}_i \cdot \nabla_i V} = \frac{n+1}{2} \bar{V}$$

True for systems with central forces

[redacted]

$$V_{ij}(r_{ij}) \sim r_{ij}^{n+1}$$

$$F_{ji} \sim r_{ij}^{n}$$

When we have only one pair or many pairs, this rule applies

For inverse square law $n = -2$

$$\overline{T} = -\frac{1}{2}\overline{V}$$

Harmonic oscillaor sytem $n = 1$

$$\overline{T} = \overline{V}$$

$$\overline{T} = -\frac{1}{2}\overline{V} \qquad\qquad \overline{T} = \overline{V}$$

Dark matter suspicion

Equipartition of energy for molecular vibrations.

Notice $\vec{r}_i \to \lambda \vec{r}_i$ $\qquad \ddot{\vec{r}}_i = \vec{F}_i$

$t \to \lambda^{+s} t$ $\qquad \lambda^{1-2s}\ddot{\vec{r}}_i = \lambda^n \vec{F}_i$

$s = \frac{1-n}{2}$ $\qquad T \to \lambda^{2(1-s)} T \to \lambda^{n+1} T$

Back to central potential

$$\dot\theta = \frac{\ell}{mr^2}$$

$$\frac{d}{dt} = \dot\theta \frac{d}{d\theta} = \frac{\ell}{mr^2}\frac{d}{d\theta}$$

$$-\frac{dV}{dr}$$

$$m\ddot r - \frac{\ell^2}{mr^3} = f(r)$$

$$\frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{mr^3} = f(r)$$

$\frac{d}{d\theta}$ $\frac{1}{\text{[redacted]}\, r^2}$ $\frac{dr}{d\theta}$ [redacted] $-$ $\frac{\text{[redacted]}}{\text{[redacted]}\, r}$ $= \frac{mr^2 f(r)}{\ell^2}$

$-\frac{d}{d\theta}\frac{du}{d\theta} -$ [redacted] $u = \frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right)$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

Example: Kepler problem

$$V(r) = -\frac{k}{r} \qquad V\left(\frac{1}{u}\right) = -ku$$

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{\ell^2}$$ [redacted] $$\frac{d(ku)}{du}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{mk}{\ell^2}\right) = 0$$

[redacted]

$$u = \frac{mk}{\ell^2} + C\cos(\theta - \theta_0)$$

$$= \frac{mk}{\ell^2}\left(1 + e\cos(\theta - \theta_0)\right)$$ [redacted]

[redacted] $$e = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$ from the following considerations:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{k}{r}$$

At max or min of r, $\dot{r} = 0$

$$\frac{1}{r_m} = \frac{mk}{\ell^2}(1 \pm e),$$

r_m could correspond to max or min.

$$\left(\frac{\ell^2}{mk\,r_m} - 1\right)^2 = e^2$$

$$1 - \frac{2\ell^2}{mk\,r_m} + \frac{\ell^4}{m^2k^2r_m^2} = e^2$$

$$1 + \frac{2\ell^2}{mk^2}\underbrace{\left(-\frac{k}{r_m} + \frac{\ell^2}{2m r_m^2}\right)}_{E} = e^2$$

So

$$1 + \frac{2E\ell^2}{mk^2} = e^2$$

Alternatively we could have done the following:

$$\theta - \theta_0 = \ell\int_0^t \frac{dt'}{mr^2(t)} = \ell\int_0^t \frac{dr}{mr^2\dot{r}}$$

$$= \ell \int_{r_0}^{r} \frac{dr}{r^2\sqrt{(E - V'(r))}}$$

$$= \frac{\ell}{\sqrt{2}m\,\text{[redacted]}} \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{E - \frac{\ell^2}{2mr'^2} - V(r')}}$$

[redacted] Using $u = \frac{1}{r}$

$$= -\frac{\ell}{\text{[redacted]}\sqrt{2m}} \int_{u_0}^{u} \frac{du'}{\sqrt{E - \frac{\ell^2}{2m}u'^2 - V\left(\frac{1}{u'}\right)}}$$

For the Kepler problem $V\left(\frac{1}{u}\right) = -ku$

$$\sqrt{E + ku - \frac{\ell^2}{2m}u^2}$$

$$= \sqrt{E + \frac{mk^2}{2\ell^2} - \frac{\ell^2}{2m}\left(u - \frac{mk}{\ell^2}\right)^2}$$

$$= \frac{\ell}{\sqrt{2m}}\sqrt{\left(1+\frac{2E\ell^2}{mk^2}\right)\frac{m^2k^2}{\text{[redacted]}\,\ell^4} - \left(u - \frac{mk}{\ell^2}\right)^2}$$

[redacted]

Appropriate transformation

$$u - \frac{mk}{l^2}[redacted] = \frac{mk}{l^2} e \cos\phi$$

With

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$ [redacted]

$$\theta - \theta_0 = [redacted] - [redacted] \int_0^{\phi} \frac{-\frac{mk}{l^2} e \sin\phi \, d\phi}{\sqrt{\frac{m^2k^2}{l^4} e^2 - \frac{m^2k^2}{l^4} e^2 \cos^2\phi}}$$

$$= \phi$$ [redacted]

$$\theta \hookrightarrow \phi$$

$$u - \frac{mk}{l^2} = \frac{mk}{l^2} e \cos(\theta - \theta_0)$$

$$\frac{1}{r} = u = \frac{mk}{l^2}\left[1 + e\cos(\theta - \theta_0)\right]$$

$$r = \frac{l^2/mk}{1 + e\cos(\theta - \theta_0)}$$

Classification of orbits

$e>1$	$E>0$	hyperbola
$e=1$	$E=0$	Parabola
$e<1$	$E<0$	ellipse
$e=0$	$E=-\frac{mk^2}{2\ell^2}$	circle

$$E = T+V$$

averaging $\quad \bar{E} = \bar{T}+\bar{V} = -\frac{\bar{V}}{2}+\bar{V} = \frac{\bar{V}}{2}$

$$E = -\overline{\frac{k}{2r}} = -\frac{k}{2}\,\overline{r^{-1}}$$

If $r=r_0$ (circle), balance between 'contrifugal' [redacted] force' and attraction

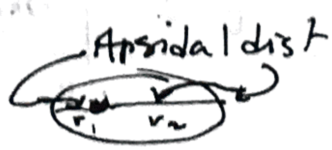
$$\frac{k}{r_0^2} = \frac{mv^2}{r_0} = \frac{\ell^2}{mr_0^3}$$

$$\Rightarrow r_0 = \frac{\ell^2}{mk} \quad \Rightarrow E = -\frac{k}{2}\overline{\frac{1}{r}}$$

$$= -\frac{k}{2}\frac{1}{r_0} = -\frac{mk^2}{2\ell^2}$$

$$e=0$$

Elliptic orbit



$$E - \frac{\ell^2}{mr^2} + \frac{k}{r} = 0 \qquad r^2 + \frac{k}{E}r - \frac{\ell^2}{2mE} = 0$$

$$r_1 + r_2 = -\frac{k}{E} \qquad \underset{\substack{\uparrow \\ \text{Semimajor axis}}}{a} = \frac{r_1+r_2}{2} = -\frac{k}{2E}$$

Note that $\quad r_{1,2} = a(1\pm e)$

$$r_1 r_2 = [redacted] -\frac{\ell^2}{2mE}$$

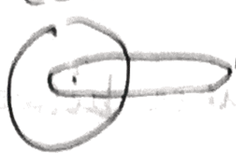
$$a^2(1-e^2) = -\frac{\ell^2}{2mk}\frac{k}{E} = \frac{\ell^2 a}{mk}$$

$$1-e^2 = \frac{\ell^2}{mka} \Rightarrow e = \sqrt{1-\frac{\ell^2}{mka}}$$

Also, using $a(1-e^2) = \frac{\ell^2}{mk}$

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_0)}$$

$e=0$



e close to 1.

Time [redacted] and angle/radius

$$t = \int_{r_0}^{r}\frac{dr}{\dot r} = \sqrt{m}\int_{r_0}^{r}\frac{dr}{\sqrt{m\dot r^2/2}} = \sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{k}{r}-\frac{\ell^2}{2mr^2}+E}}$$

Also using $dt = \frac{mr^2}{\ell}d\theta$

$$t = \int_{\theta_0}^{\theta}\frac{mr^2 d\theta}{\ell} = \frac{\ell^3}{mk^2}\int_{\theta_0}^{\theta}\frac{d\theta}{[1+e\cos(\theta-\theta')]^2}$$

Sometimes it is better to express things in terms of eccentric anomaly

$$r = a(1-e\cos\psi)$$

$$\sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{\frac{k}{r} - \frac{l^2}{2mr^2} + E}} = \sqrt{\frac{m}{2}} \int \frac{r\,dr}{\sqrt{Er^2 + kr - \frac{l^2}{2m}}}$$

$$= \sqrt{\frac{m}{2}} \int \frac{r\,dr}{\sqrt{E\left(r^2 + \underbrace{\frac{k}{E}}_{-2a} r - \underbrace{\frac{l^2}{2mE}}_{a^2(1-e^2)}\right)}}$$

$$E = -\frac{k}{2a}$$

$$= \sqrt{\frac{m}{2k}} \int \frac{r\,dr}{\sqrt{-\frac{1}{2a}\left(r^2 - 2ar + a^2(1-e^2)\right)}} = \sqrt{\frac{ma}{k}} \int \frac{r\,dr}{\sqrt{a^2e^2 - (r-a)^2}}$$

$$r = a(1 - e\cos\psi)$$

$$t = \sqrt{\frac{ma}{k}} \int \frac{a^2(1 - e\cos\psi)\sin\psi\,d\psi}{\sqrt{a^2e^2(1 - \cos^2\psi)}} = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1 - e\cos\psi)\,d\psi$$

Note that $\underbrace{T}_{\text{Period}} = 2\pi \sqrt{\frac{ma^3}{k}}$ from integrating around the orbit.

Laplace-Runge-Lenz Vector

$$\dot{\vec{p}} = f(r)\,\frac{\vec{r}}{r}$$

$$\dot{\vec{p}} \times \vec{L} = m f(r)\, \frac{\vec{r}}{r} \times (\vec{r} \times \dot{\vec{r}})$$

$$= \frac{m f(r)}{r}\left[\vec{r}(\vec{r}\cdot\dot{\vec{r}}) - r^2 \dot{\vec{r}}\right]$$

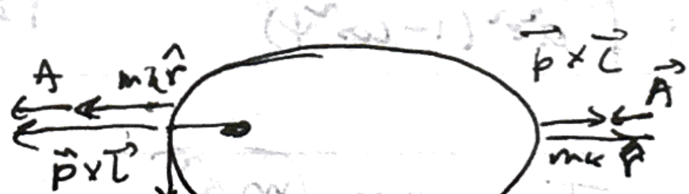
$$= \frac{m f(r)}{r}\left[\vec{r}\, r\dot{r} - r^2 \dot{\vec{r}}\right]$$

$\vec{L}$ is constant, so $\frac{d}{dt}(\vec{p} \times \vec{L}) = -m f(r) r^2 \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$

For the kepler problem $\quad f(r) = -\frac{k}{r^2}$

$$\frac{d}{dt}\left(\vec{p} \times \vec{L} - \frac{mk\vec{r}}{r}\right) = 0$$

$$\frac{d}{dt}\vec{A} = 0$$



$$\vec{A} = \vec{p} \times \vec{L} - \frac{mk\vec{r}}{r}$$

$$\vec{A}\cdot\vec{r} = \vec{r}\cdot(\vec{p} \times \vec{L}) - mkr$$

$$= \vec{L}\cdot(\vec{r} \times \vec{p}) - mkr$$

$$= L^2 - mkr$$

$$A r\cos\theta = \ell^2 - mkr$$

$$\frac{1}{r} = \frac{mk}{\ell^2}\left(1+\underbrace{\frac{A}{mk}}_{e}\ \text{[redacted]}\ \cos\theta\right)$$

[redacted] $A = mke$

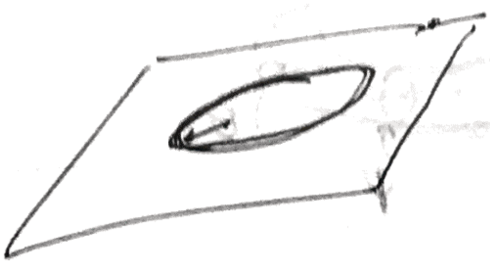
$\vec{A}$ is not completely independent of E & $\vec{L}$.

$$A^2 = m^2k^2e^2 = m^2k^2\left(1+\frac{2E\ell^2}{mk^2}\right)$$
$$= m^2k^2 + 2mE\ell^2$$

$E, \vec{L} \to$ 4 conserved quantities

$\vec{A}$: adds one more conserved quantity

$\Rightarrow$ angle of perihelion



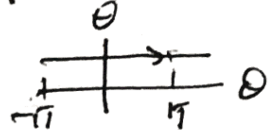
Existence of such extra [redacted] conserved quantity has something to do with having closed orbits [redacted] in the Kepler problem

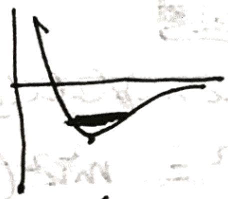
Consider perturbations of orbits around a circular one.

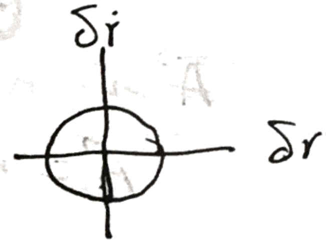
Focus on 2D

Angle: 

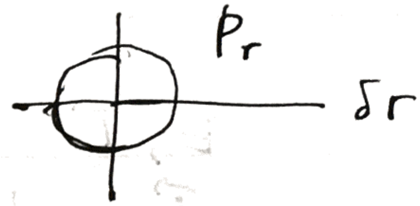
$\dot{\theta} \approx \frac{l}{m r_0^2} = \omega_\theta$ 

$\theta \sim \omega_0 t$ $\quad p_\theta = l$ constant 

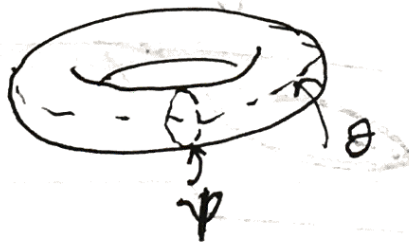
Radial 



$\delta r = r - r_0$ $\quad \delta r = a e \cos\psi(t)$

$E - E_0 =$ constant 

$\psi(t) \approx \omega_r t$



In general ω_θ and ω_r are not commensurate

To have closed orbits for generic initial conditions:

Firstorder analysis suggests

$f(r) = -\frac{k}{r^{3-\beta^2}}$ with β rational

Higher order analysis $\Rightarrow \beta^2 = 1, 4$

Bertrand's Theorem.

Additional conserved quantities

Foliation of the ~~tota~~ torus

[Circular orbit: $\omega_\theta = \frac{\ell}{m r_0^2}$ $\qquad f(r_0) + \frac{\ell^2}{m r_0^3} = 0$

Linearize $\quad m\,\ddot{\delta r} = \left(f'(r_0) - \frac{3\ell^2}{m r_0^4}\right)\delta r$

$$= -\left(3 - \frac{m r_0^4 f'(r_0)}{\ell^2}\right)\omega_\theta^2\,\delta r$$

$$= -\left(3 + \frac{r_0 f'(r_0)}{f(r_0)}\right)\omega_\theta^2\,\delta r$$

$$= -\beta^2(r_0)\,\omega_\theta^2\,\delta r$$

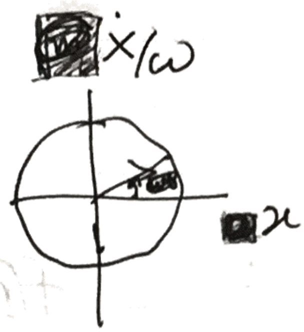
β cont fnc of r_0 and rational $\Rightarrow$ const.

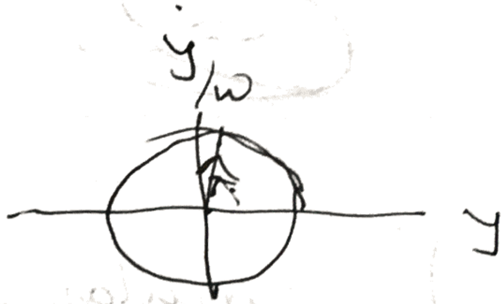
$$\frac{r_0 f'(r)}{f(r_0)} = \beta^2 - 3 \quad \Rightarrow \quad f(r) = -\frac{k}{r^{3-\beta^2}}$$

with β rational]

Additional conserved quantities for isotropic harmonic oscillator

$$\frac{m}{2}\dot{x}_i\dot{x}_j + \frac{m\omega^2}{2}x_i x_j$$

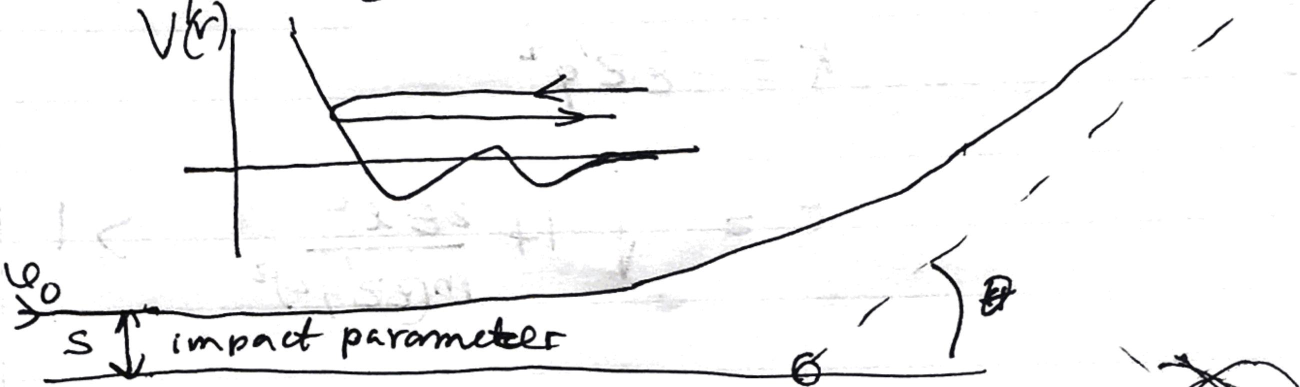




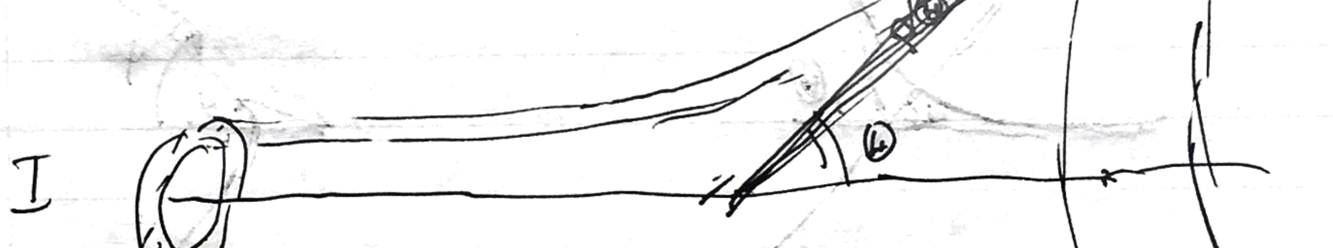
[redacted]

Angle between these phase space vectors are preserved.

Scattering in a central force



$$l = m v_0 s = s\sqrt{2mE}$$



$$2\pi I s\, ds = 2\pi I \sigma(\Theta) \sin\Theta\, d\Theta$$

↑ area unit

Express $s = s(\Theta, E)$

$$\sigma(\Theta) = \frac{s}{\sin\Theta} \frac{ds}{d\Theta}$$

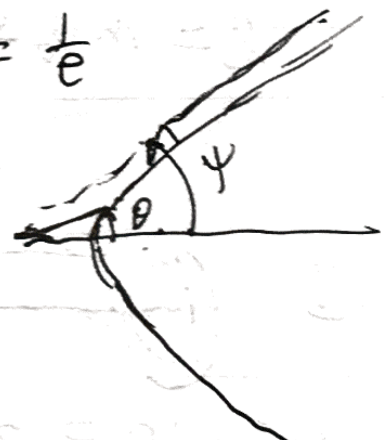
Application to Coulomb Repulsion

$$k = -ZZ'q^2$$

$$e = \sqrt{1 + \frac{2E\ell^2}{m(ZZ'q^2)^2}} > 1$$

$$\frac{1}{r} = \frac{mZZ'q^2}{\ell^2}(e\cos\theta - 1)$$

$$\cos\psi = \frac{1}{e}$$



$$\Theta = \pi - 2\psi$$

$$\psi = \frac{\pi}{2} - \frac{\Theta}{2} \qquad \sin\frac{\Theta}{2} = \frac{1}{e(s,E)}$$

$$1 + \frac{2E\ell^2}{m(ZZ'q^2)^2} = e^2 = \csc^2\frac{\Theta}{2} = 1 + \cot^2\frac{\Theta}{2}$$

$$s = \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}$$

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{ZZ'q^2}{2E}\right)^2 \csc^4\frac{\Theta}{2}$$

Divergence at $\Theta \to 0$
Long range interaction

Rutherford Scattering: Large back scattering