Conservation Theorems and Symmetries [redacted]

$\frac{\partial L}{\partial \dot{q}_j}$ [redacted] could be interpreted as momenta.

$$L = \frac{1}{2}\sum_i m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V$$ [redacted]

no velocities

$$\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_{ix}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

canonical/Conjugate momentum.

Particles in an E.M. field

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \sum_i q_i \phi(\vec{r}_i) + \sum_i q_i \text{[redacted]} \vec{A}(\vec{r}_i)\cdot\dot{\vec{r}}_i$$

$$\vec{p}_i = m_i \dot{\vec{r}}_i + q_i \vec{A}(\vec{r}_i)$$

If L is independent of q_j

$$\dot{p}_j = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0$$

q_j is called cyclic / ignorable

The corresponding momentum is conserved:

p_j = constant : ~~A~~ first Integral of motion.

$p_j(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \text{Const.}$

In principle, one could try to transform to coordinates so that "all" coordinates become cyclic. It works out for 'integrable' systems.

More on it later.

For conservative systems, with V fnc of coordinates only

$$\dot{p}_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Recall $\dfrac{\partial \vec{r}_i}{\partial q_j} = \dfrac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \dfrac{\partial \vec{v}_i}{\partial \dot{q}_j}$

~~Remember~~

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i \vec{\nabla}_{\vec{v}_i} T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Two examples $\qquad Q = \left(\sum \vec{F}_i\right)\cdot \hat{n}$

Translation $\qquad \vec{r}_i \to \vec{r}_i + q\hat{n}$

$$\frac{\partial r_i}{\partial q} = \hat{n}$$

$$p = \left(\sum m_i \vec{v}_i\right)\cdot \hat{n} = \hat{n}\cdot \vec{p}$$

Rotation

$d\vec{r}_i$

$= (\hat{n}\, d\theta) \times \vec{r}_i$

$$Q = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial \theta} = \vec{F}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \hat{n} \cdot \left(\sum_i \vec{r}_i \times \vec{f}_i \right) = \hat{n} \cdot \vec{N}$$

Component of torque in $\hat{n}$'s direction

$$p_\theta = \sum_i m_i \vec{v}_i (\hat{n} \times \vec{r}_i)$$

$$= \hat{n} \cdot \sum m_i \vec{r}_i \times \vec{v}_i$$

$$= \hat{n} \cdot \vec{L}$$

Conservation laws result of symmetry

$L = \frac{1}{2} \sum_i m_i v_i^2 + \sum_{i<j} V_{ij}(\vec{r}_{ij}) \Rightarrow \vec{p}$ conserved

$L = \frac{1}{2} \sum_i m_i \vec{v}_i^2 + \sum_i V_i(r_i) + \sum_{i<j} V_{ij}(r_{ij}) \Rightarrow \vec{L}$ conserved

Energy conservation and time-translation invariance

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right) + \frac{\partial L}{\partial t}$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right] + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt} \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt} h = -\frac{\partial L}{\partial t}$$

$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$ is the

Hamiltonial / Energy function

If L has no explicit time dependence

h = constant. (Jacobi's integral)

Remember Rayleigh dissipation function?

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot q_j}$$

Same derivation $\quad \frac{dh}{dt} = -\frac{\partial L}{\partial t} - \sum_j \frac{\partial \mathcal{F}}{\partial \dot q_j}\dot q_j$

If $\mathcal{F}$ is a homogeneous fnc of of degree 2 in $\dot q$'s

[Example: $\sum_{ij} f_{ij}(q)\,\dot q_i \dot q_j$, but also

$$\frac{\sum f_{ijkl}\,\dot q_i \dot q_j \dot q_k \dot q_l}{\sum g_{ij}\,\dot q_i \dot q_j}$$]

then $\quad \sum_j \frac{\partial \mathcal{F}}{\partial \dot q_j}\dot q_j = 2\mathcal{F}$

So $\quad \frac{dh}{dt} = -2\mathcal{F} - \frac{\partial L}{\partial t}$

When $\frac{\partial L}{\partial t} = 0$

$$\frac{dh}{dt} = -2\mathcal{F}$$

[redacted] Change in energy

$$= -2\int_1^2 \mathcal{F}\, dt$$

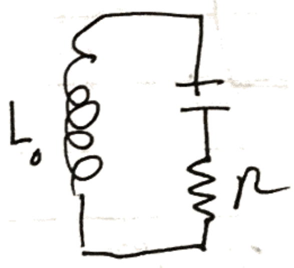
Interesting example: LCR circuits

$q \to$ charge $\quad i = \dot{q}$

$$L = \frac{L_0 \dot{q}^2}{2} \text{ [redacted] } - \frac{q^2}{2C}$$

Contr. from Inductance (like K.E.)

Contr. from Capacitance (like P.E.)



$$\mathcal{F} = \frac{1}{2} R \dot{q}^2$$