Back to Lagranges equation

$$\delta \int_1^2 L(q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_n, t)\, dt$$

$$= 0$$

$$q_i(t) \to q_i(t) + \alpha\, \eta_i(t)$$

$$\eta_i(t_{1,2}) = 0$$

$$\frac{d}{d\alpha}\int L\, dt = \sum_j \int_1^2 \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_j}\right)\eta_j(t)\, dt$$

Choosing individual η's to be like 

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot q_j}$$

Ex $\quad L = \frac{m}{2}\dot x^2 - V(x)$

Dealing with additional constraints:

$$f_a(q_1, \dots, q_n, t) = 0$$

$$\int_1^2 \left[L(q_1, \dots, \dot{q}_1, \dots, t) + \sum_a \lambda_a(t) f_a(q_i, t) \right] dt$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \underbrace{\sum_a \lambda_a \frac{\partial f_a}{\partial q_j}}_{Q_j} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} + Q_j$$

forces due to constraint

$$L = \frac{1}{2} M(\dot{x}^2 + \dot{z}^2) - Mgz$$

$x = r\sin\theta,\ z = r\cos\theta$

$$\sqrt{x^2 + z^2} = a$$

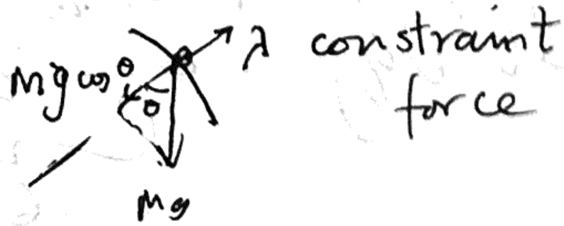
$$L = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mgr\cos\theta$$

Constraint $\boxed{r - a = 0}$ $\qquad \delta\int [L + \lambda(t)(r-a)]\,dt = 0$

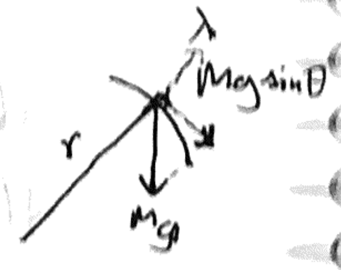
[redacted]

Equation of motion

$$\boxed{M\ddot{r} = Mr\dot{\theta}^2 - Mg\cos\theta + \lambda(t)}$$



$$\boxed{\frac{d}{dt}(Mr^2\dot{\theta}) = Mgr\sin\theta}$$



Three equations, including the constraint equation, determining $r(t)$, $\theta(t)$, $\lambda(t)$

Now we use $r(t) = a$

$$0 = Ma\dot{\theta}^2 - Mg\cos\theta + \lambda$$
$$Ma^2\ddot{\theta} = Mga\sin\theta$$

Second equation $\Rightarrow \frac{Ma^2}{2}\dot{\theta}^2 + Mga\cos\theta = E$ constant

Initially $\dot{\theta} = 0$ $\cos\theta = 1$

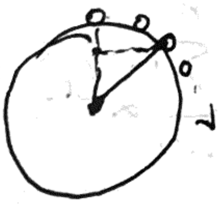
$$\Rightarrow \quad \ddot{\theta}^2 = \frac{2g}{a}(1-\cos\theta)$$

Use that in the first equation

$$\lambda = Mg\cos\theta - Ma\dot{\theta}^2$$

$$= Mg\cos\theta - Ma\frac{2g}{a}(1-\cos\theta)$$

$$= Mg(3\cos\theta - 2)$$



$\lambda = 0$ when $\cos\theta = \frac{2}{3}$

when $\cos\theta < \frac{2}{3}$ $\lambda < 0$.

$\Rightarrow$ That is not possible in this case [non-holonomic nature of constraint]

$\Rightarrow$ Ball leaves constraint surface.

Semi-holonomic Constraints

$$f_\alpha(q_1, \cdots, q_n; \dot q_1, \cdots, \dot q_n; t) = 0$$

$\alpha = 1, \cdots, m$.

Let us take a special case:

$$f_\alpha = \sum_{k=1}^{n} a_{\alpha k}(t)\, \dot q_k + a_{\alpha 0}(t) = 0$$

$$\delta \int_{t_1}^{t_2} \left[L + \sum_{\alpha=1}^{m} \mu_\alpha(t) f_\alpha \right] dt$$

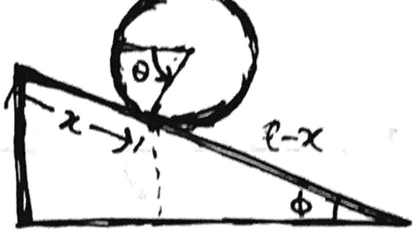
$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{d}{dt}\sum_{\alpha=1}^{m} \mu_\alpha(t)\, a_{\alpha k}(t) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = \underbrace{\frac{d}{dt}\sum_{\alpha=1}^{m} \mu_\alpha a_{\alpha k}}_{Q_k^c}$$

$$\left(= \sum_{\alpha=1}^{m} \frac{d}{dt}\left(\mu_\alpha \frac{\partial f_\alpha}{\partial \dot q_k}\right)\right)$$

in this case

Generalization for $f_\alpha(q_1, \cdots, q_n, \dot q_1, \cdots, \dot q_n, t)$ possible.



$dx = r\,d\theta$

$\dot{x} = r\dot{\theta}$

r is constant.

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mr^2\dot{\theta}^2$$

$$V = Mg(l-x)\sin\phi$$

$$f = r\dot{\theta} - \dot{x} = 0 \qquad a_\theta = r \quad a_x = -1$$

$$\delta\int_1^2 dt\left[\frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mr^2\dot{\theta}^2 - Mg(l-x)\sin\phi + \mu(r\dot{\theta}-\dot{x})\right] = 0$$

$M\ddot{x} - Mg\sin\phi -$ [redacted] $\dot{\mu} = 0$

$$Mr^2\ddot{\theta} + \dot{\mu}r = 0$$

Let us call $(-\dot{\mu})$ [redacted] λ.

$$\boxed{\begin{aligned} M\ddot{x} &= Mg\sin\phi - \lambda \\ Mr^2\ddot{\theta} &= \lambda r \end{aligned}}$$

λ is the force applied by the plane.
Force & torque equations.

These two, along with

$$\boxed{r\dot{\theta} = \dot{x}}$$

are the three equations that determine $x(t)$, $\theta(t)$, $\lambda(t)$

$$r\dot{\theta} = \dot{x} \Rightarrow r\ddot{\theta} = \ddot{x}$$

$Mr^2\ddot{\theta} = \lambda r$ then says $M\ddot{x} = \lambda$

Using it in $M\ddot{x} = mg\sin\phi - \lambda$

$$\ddot{x} = \frac{g\sin\phi}{2}$$

$$\lambda = \frac{Mg\sin\phi}{2}$$

$$\ddot{\theta} = \frac{g\sin\phi}{2r}$$

constants

Of course this is a very fancy way of solving a simpler problem. In this particular case,

We could have integrated $\dot{x} = r\dot{\theta}$ and eliminate $\dot{\theta}$.
(That is not possible, in general, for semi holonomic constraints).

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M r^2 \dot{\theta}^2 = M \dot{x}^2$$

$$\delta \int [M \dot{x}^2 - Mg(l - x) \sin\phi] dt$$

$$2M\ddot{x} = Mg \sin\phi$$

$$\ddot{x} = \frac{g \sin\phi}{2}$$