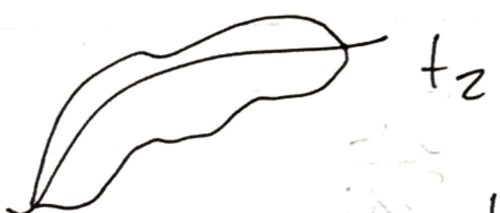
Variational Principle

Hamilton's Principle: Action integral is stationary on the actual path.



Paths in configuration space.

t_1

$$I = \int_{t_1}^{t_2} L\,dt$$

Action integral

[Monogenic: All applied force comes from a generalized scalar potential]

In other words:

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,dt$$

$$= 0$$

Comments on least vs. stationary

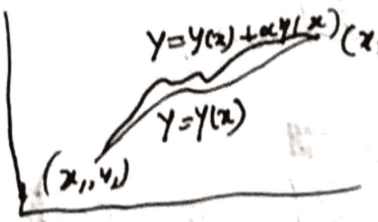
Calculus of Variations

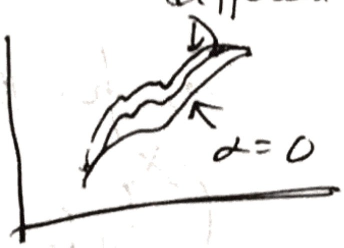
One dim example

$f(y, \dot{y}, x)$ defined on a path $y = y(x)$

$$\dot{y} = \frac{dy}{dx}$$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx$$





Deformed path $\quad y(x, \alpha) = y(x, 0) + \alpha \eta(x)$

$\uparrow$
$y(x)$

$\eta(x_1) = \eta(x_2) = 0$

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)$$

Stationary path condition

$$\left(\frac{d J(\alpha)}{d\alpha}\right)_{\alpha = 0} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} \right) dx$$

Notice that $\dfrac{\partial \dot{y}}{\partial \alpha} = \dfrac{\partial^2 y(x,\alpha)}{\partial \alpha \partial x} = \dfrac{\partial^2 y(x,\alpha)}{\partial x \partial \alpha}$

So, for the second term,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \alpha}\right) dx$$

$$= \left. \frac{\partial f}{\partial \dot{y}} \frac{\partial y}{\partial \alpha} \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \dot{y}}\right) \frac{\partial y}{\partial \alpha} dx \quad \Big\} \text{ integration by parts}$$

If the varied curve passes through the boundary points $\Rightarrow \dfrac{\partial y}{\partial \alpha} = \eta = 0$ at x_1, x_2.

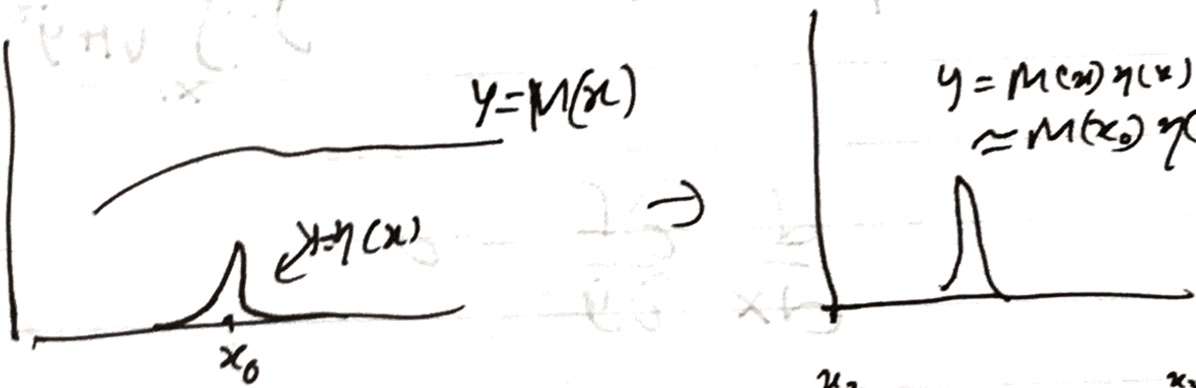
Thus $\displaystyle \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right)\right]\frac{\partial y}{\partial \alpha}\, dx$

going to $\alpha = 0$

$\frac{\partial y}{\partial \alpha}$

$$\int_{x_1}^{x} M(x)\,\eta(x)\,dx = 0 \qquad M(x) = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot y}$$

for arbitrary "smooth" η.



$$\int_{x_1}^{x_2} M\,\eta\,dx \cong M(x_0)\int_{x_0} \eta(x)\,dx$$

$= 0$

$M(x_0) = 0$ for every x_0

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot y} = 0$$

Usual notation $\quad \frac{\partial y}{\partial \alpha}\delta\alpha = \delta y$

$$\delta J = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot y}\right)\delta y\, dx$$

Examples:

Shortest distance between two point

$$ds^2 = \sqrt{dx^2+dy^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\dot{y}^2}\,dx$$

$$f = \sqrt{1+\dot{y}^2} \qquad J = \int_{x_1}^{x_2} \sqrt{1+\dot{y}^2}\,dx$$

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

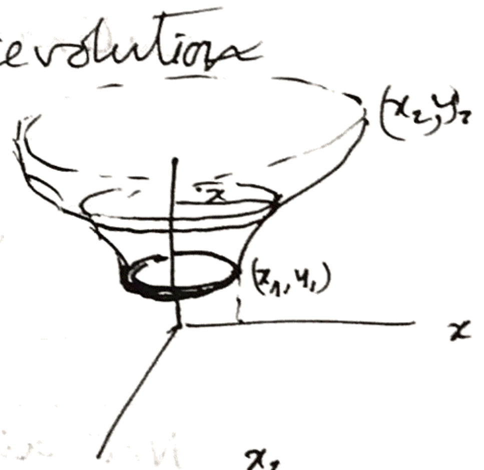
$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c \Rightarrow \dot{y} = a$$

$$a = \frac{c}{\sqrt{1-c^2}}$$

$$y = ax + b$$

Straight line

Minimum Surface of revolution



$$\text{Area} = \int_{x_1}^{x_2} 2\pi x \, ds = 2\pi \int_{x}^{x_2} x\sqrt{1+\dot{y}^2}\, dx$$

$$f = x\sqrt{1+\dot{y}^2} \qquad \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a$$

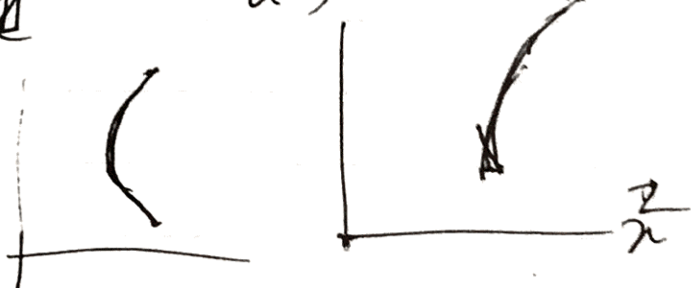
$$(x^2 - a^2)\dot{y}^2 = a^2 \quad [redacted]$$

$$y = \int \frac{dy}{dx}\, dx = \int \frac{a\, dx}{\sqrt{x^2 - a^2}}$$

$$x = a \cosh t \qquad \int \frac{a \sinh t \, dt}{a \sinh t} = a[redacted] t + b$$

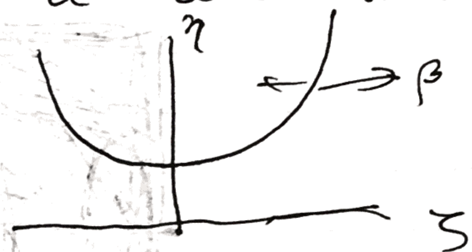
$$x = a\cosh t = [redacted] \; a \cosh \frac{y-b}{a},$$

Even b like →



Catenary: Define $\frac{x}{a}=\xi$ $\frac{y}{a}=\eta$ $\frac{b}{a}=\beta$

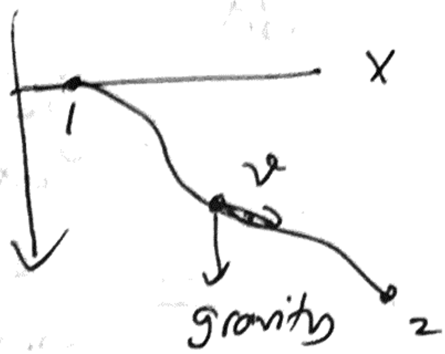
$$\xi = ch(\eta-\beta)$$



Not all [redacted] initial and final conditions lead to unique [redacted] [redacted] solutions. Some lead to none.

The brachistochrone problem

t_{12}



$$t_{12} = \int \frac{ds}{v}$$

$$\frac{1}{2} m v^2 = m g y$$

$$v = \sqrt{2gy}$$

$$t_{12} = \int \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\, dx$$

$$f = \sqrt{\frac{1+\dot{y}^2}{2gy}}$$

$\frac{\partial f}{\partial y} = -\frac{1}{2y}$ [redacted] $\sqrt{\frac{1+\dot{y}^2}{2gy}}$ $\qquad$ $\frac{\partial f}{\partial \dot{y}} = \frac{1}{\sqrt{2gy}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$

$\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial \dot{y}}$ looks a bit complicated.

If $f = f(y, \dot{y})$, no x dependence,

$\frac{df}{dx} = \dot{y} \frac{\partial f}{\partial y} + \ddot{y} \frac{\partial f}{\partial \dot{y}} + \cancel{\frac{\partial f}{\partial x}}$ ([redacted] $\frac{\partial f}{\partial x} = 0$)

$$\frac{df}{dx} = \dot{y}\frac{\partial f}{\partial y} + \ddot{y}\frac{\partial f}{\partial \dot{y}}$$

$$= \dot{y}\frac{d\,\partial f}{dx\,\partial \dot{y}} + \ddot{y}\frac{\partial f}{\partial \dot{y}}$$

$$= \frac{d}{dx}\left(\dot{y}\frac{\partial f}{\partial \dot{y}}\right)$$

That means $\frac{d}{dx}\left(\dot{y}\frac{\partial f}{\partial \dot{y}} - f\right) = 0$

or $\dot{y}\frac{\partial f}{\partial \dot{y}} - f = \text{constant}.$

[Aside: $L = \frac{m}{2}\dot{x}^2 - V(x)$

$$\dot{x}\frac{\partial L}{\partial \dot{x}} - L = m\dot{x}^2 - \left\{\frac{m\dot{x}^2}{2} - V(x)\right\}$$

$$= \frac{1}{2}m\dot{x}^2 + V(x)$$

Energy = constant]

Using it for the brachistochrone problem:

$$\frac{\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} - \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

$$= \frac{\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} - \frac{1+\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}}$$

$$= -\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} \rightarrow \text{constant}$$

$$\sqrt{y}\sqrt{1+\dot{y}^2} = k$$

$$\frac{dy}{dx} = \sqrt{\frac{k^2-y}{y}}$$

$$\int dx = \int \sqrt{\frac{y}{k^2-y}}\, dy$$

Natural transformation

$$y = k^2 \sin^2\theta$$

$$\int \sqrt{\frac{y}{k^2-y}}\, dy = 2k^2 \int \sin^2\theta\, d\theta$$

$$= k^2 \int (1 - \cos 2\theta)\, d\theta$$

$$= k^2\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

Use $2\theta = \phi$ and $\frac{k^2}{2} = a$ to get

$$x = a(\phi - \sin\phi)$$

$$y = a(1 - \cos\phi)$$

We used the initial condition $x=0, y=0$ to set $C=0$.

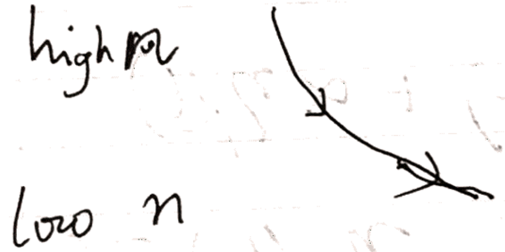
Cycloid

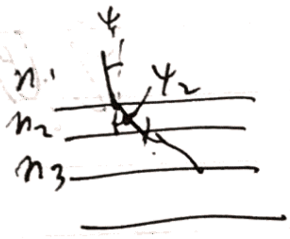
What is this 'conserved' quantity

$$\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} \quad ?$$

Analogy to Fermat's Principle and Snell's law.

$$\int_1^2 n\,ds \quad \leftrightarrow \quad \int_1^2 \frac{1}{v}\,ds$$





$$n_1 \sin\psi_1 = n_2 \sin\psi_2 = n_3 \sin\psi_3 = \dots$$

$n \sin\psi$ is 'conserved'.

$$\tan\left(\frac{\pi}{2}-\psi\right) = \dot{y}$$

$$\sqrt{1+\dot{y}^2} = \sqrt{\sec^2\left(\frac{\pi}{2}-\psi\right)} = \sec\left(\frac{\pi}{2}-\psi\right)$$

$$= \frac{1}{\sin\psi}$$

$$n\sin\psi \rightarrow \frac{\sin\psi}{v} = \frac{1}{\sqrt{2gy}\,\sqrt{1+\dot{y}^2}}$$