Applications:

~~Examples~~

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \sum \frac{1}{2} m_i \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)^2$$

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2}\sum_{jk} M_{jk} \dot{q}_j \dot{q}_k$$

$$\frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \qquad \sum_i m_i \frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

Examples:

Cartesian coordinates

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \text{etc.}$$

$$\frac{d}{dt}(m\dot{x}) = F_x \quad \text{etc.}$$

Polar coordinates in 2D:

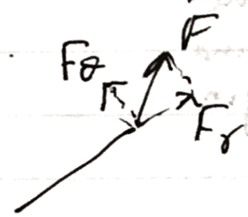
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = Q_r$$

$$Q_r = F_x\frac{\partial x}{\partial r} + F_y\frac{\partial y}{\partial r}$$
$$= F_x\cos\theta + F_y\sin\theta$$
$$= F_r$$



$$m\ddot{r} - mr\dot{\theta}^2 = F_r$$

centripetal acceleration

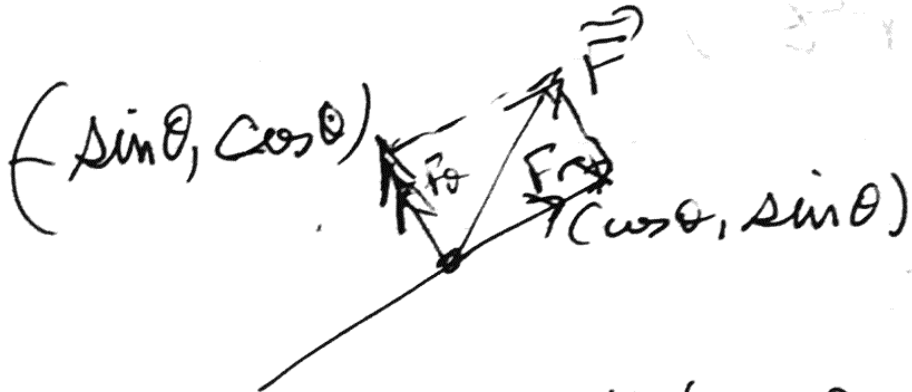
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}\left(m r^2 \dot\theta\right) - 0 = Q_\theta$$

$$Q_\theta = F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta}$$

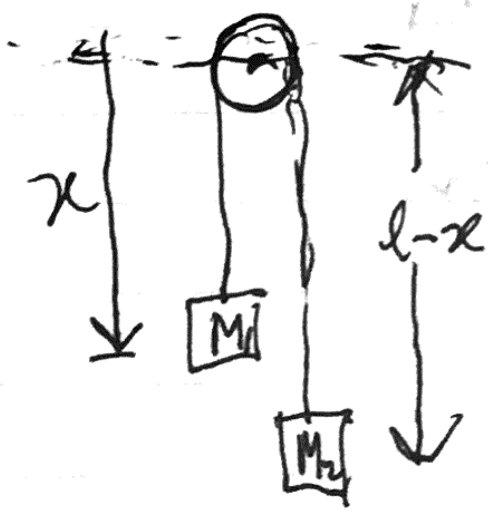
$$= -F_x\, r \sin\theta + F_y\, r\cos\theta$$

$$= r F_\theta$$



$$\underbrace{\frac{d}{dt}\left(m r^2 \dot\theta\right)}_{\text{angular momentum}} = \underbrace{r F_\theta}_{\text{torque around origin}}$$

Atwood's Machine



$$T = \frac{1}{2}(M_1+M_2)\,\dot{x}^2$$

$$V = -M_1 g x - M_2 g(l-x)$$

$$L = T - V$$

$$= \frac{1}{2}(M_1+M_2)\,\dot{x}^2 + M_1 g x + M_2 g(l-x)$$

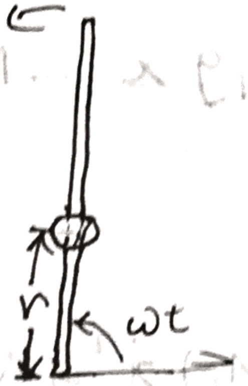
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$(M_1+M_2)\,\ddot{x} = (M_1-M_2)\,g$$

$$\ddot{x} = \frac{M_1-M_2}{M_1+M_2}\,g$$

The machine could be used for demonstrating constant acceleration motion, controlling the acceleration by changing the masses.

A bead sliding on a uniformly rotating wire in a force-free space



$x = r\cos\omega t$

$y = r\sin\omega t$

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$m\ddot{r} = mr\omega^2$$

Centrifugal force

$$\ddot{r} = \omega^2 r \Rightarrow r(t) \sim e^{\omega t}$$

Angular momentum L_z

$$mr^2\omega \sim mr_0^2\omega e^{2\omega t}$$

Not conserved

Constraint force $= \frac{\dot{L}}{r} = 2m\dot{r}\omega^2$

An aside:

Note on charged particle in EM field

$$L = \frac{1}{2} m v^2 - q\phi + q\vec{v}\cdot\vec{A}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \qquad i=1,2,3$$

$$\partial_i = \frac{\partial}{\partial x_i} \qquad \partial_t = \frac{\partial}{\partial t}$$

$$\frac{d}{dt}\left(m v_i + q A_i\right) = -q\,\partial_i \phi + q\, v_j \partial_i A_j$$

$$m\dot{v}_i = -q\,\partial_i\phi + q\, v_j \partial_i A_j$$

$$- q\left[v_j \partial_j A_i + \partial_t A_i \right]$$

$$= q\left[\left(-\partial_i\phi - \partial_t A\right) + \left(v_j \partial_i A_j - v_j \partial_j A_i\right)\right]$$

$$= q\Big[\underbrace{-\left(\partial_i\phi + \partial_t A\right)}_{-\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} = \vec{E}} + \underbrace{\varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m}_{\vec{v}\times(\vec{\nabla}\times\vec{A}) = \vec{v}\times\vec{B}}\Big]$$