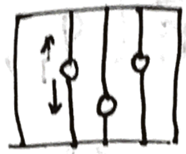
Holonomic Constraints

Some cases we do not know enough about forces but know the consequences of the forces.

Constraints :



Abacus beads

Gas molecule constrained by wall

Holonomic

Non-holonomic

Constraints expressed as

$$f(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0$$

Rigid body $\quad |\vec{r}_i - \vec{r}_j|^2 - c_{ij}^2 = 0$

Holonomic

Rheonomous $\quad f(\vec{r}_1, \vec{r}_2, \ldots, t) = 0$

Explicit time dependence

Scleronomous $\quad f(\vec{r}_1, \ldots, \vec{r}_N) = 0$

Time independent

Examples of non-holonomic constraint:

$r^2 - a^2 \geq 0$

Inequality

↳ Non-holonomic

Other kind where velocities are related :

rolling without slipping

Problems : $\quad \vec{r}_i$ not independent anymore

$\vec{F}_i$ not completely known

Problem 1 → Use generalized coordinates

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_{3N-k}, t)$$

N particles → 3N coordinates
k holonomic constraints

3N−k independent variables.

Example: Rigid body

3 angles describe the orientation of rigid body
3 coordinates describe its center of mass

For any particle

$$\vec{r}_i = \vec{r}_i(\underbrace{X, Y, Z}_{\text{C.M. coordinates}}, \underbrace{\phi, \theta, \psi}_{\text{Eulerian angles}})$$

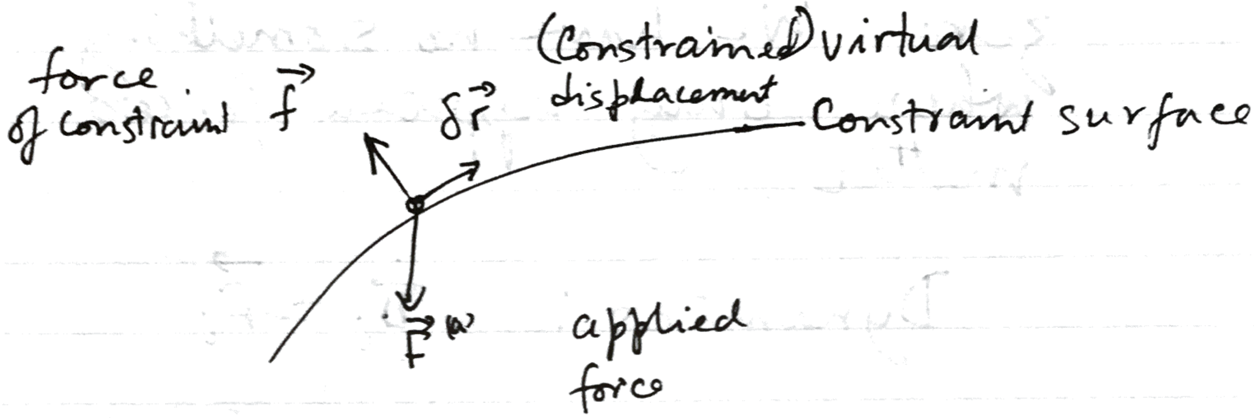
What about Problem 2: Forces?

We would like the forces due to constraints not to be invoked explicitly.

[illegible] $r_{ij} = c_{ij}$

Hint! Work done by constraint maintaining forces would be zero.

D'Alembert's Principle



The key idea is that

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i$$

total force

This is zero in many cases.
(Violated if there is friction)

For static equilibrium

$$\vec{F}_i = 0 \Rightarrow \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$$

$$\Rightarrow \sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$$

Note that $\vec{f}_i^{(o)}$ need not be zero. We have no something where only applied forces matter.

Dynamics: $\dot{\vec{p}}_i = \vec{F}_i$

$$\Rightarrow \sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

$$\Rightarrow \boxed{\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0}$$

D'Alembert's Principle

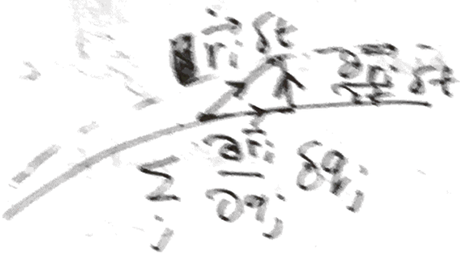
Since $\delta \vec{r}_i$'s are not all independent, we need to go back to generalized coordinates q to decouple them and derive equations of motion.

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, t)$$

One has to be a bit careful ~~when~~ with the time dependence of this transformation

$$\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

$$= \vec{v}_i(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots, t)$$



Note that

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

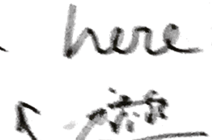
Also $\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial q_j}$

Now we are ready!

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \vec{F}_i^{(a)} \cdot \delta r_i$$

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

Note that we have no $\frac{\partial r_i}{\partial t}\delta t$ term here.



Force of constraint does some work with changing constraint!

Ex. Pendulum w/ changing length

$$\sum_{ij} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{ij} \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

Since δq_j's are independent
we can set all but one to zero

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \underbrace{\sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}}_{Q_j \text{ generalized force}}$$

$$\sum_i \frac{d}{dt}\left(m\vec{v}_i\right) \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= \sum_i \left[\frac{d}{dt}\left(m\vec{v}_i \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{\frac{\partial \vec{v}_i}{\partial \dot{q}_j}}\right) - m\vec{v}_i \cdot \underbrace{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)}_{\frac{\partial \vec{v}_i}{\partial q_j}}\right]$$

$$= \sum_i \left[\frac{d}{dt}\left(m\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]$$

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$$

[redacted] $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

[redacted] If $\vec{F}_i^{(a)} = -\vec{\nabla}_i V$

$$Q_j = \sum_i \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \vec{\nabla}_i V$$

$$V(\vec{r}_1, \vec{r}_2, \dots, t)$$
$$= V(\vec{r}_1(q_1, q_2, \dots, t), \vec{r}_2(q_1, q_2, \dots, t), \dots, t)$$

$$\frac{\partial V}{\partial q_j} = \sum_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \vec{\nabla}_i V$$

So [redacted] $Q_j = -\dfrac{\partial V}{\partial q_j}$

Define $L = T - V$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$ Lagrange's Equation.

Comment 1

In [redacted] general, we could have

$$L' = L + \frac{dF(q, [redacted] t)}{dt}$$

and it will produce the same equations of motion.

$$\dot{F} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$$

$$\frac{\partial \dot{F}}{\partial q_j} = \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j}$$

Equal

$$\frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)$$

$$= [redacted] \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 F}{\partial t \partial q_i}$$

So L is not unique.

Comment 2

$$L = T - U$$ where $U(q, \dot{q})$ is possible

Example: Particle in an EM field

$$m\dot{\vec{V}} = \vec{F} = q\left[\vec{E} + \underbrace{\vec{V} \times \vec{B}}\right]$$

How to get the velocity dependent term?

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$U = q\phi - q\vec{A}\cdot\vec{v}$$

$$L = \frac{1}{2}m\dot{x}^2 - q\phi(x) + q\,\vec{A}(x,t)\cdot\dot{\vec{x}}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$$

$$\frac{\partial L}{\partial x} = -q\frac{\partial \phi}{\partial x} + q\dot{\vec{x}}\cdot\frac{\partial \vec{A}}{\partial x}$$

$$= -q\frac{\partial \phi}{\partial x} + q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[m\dot{x} + q A_x\right]$$

$$= m\ddot{x} + q\left[\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}\right]$$

$$m\ddot{x} = q\Big[-\frac{\partial \phi}{\partial x}\frac{\partial A_x}{\partial t} + \left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$-\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}\right)\Big]$$

$$= q\Big[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}$$

$$+\left\{\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\}\Big]$$

$$= q\left[E_x + \left(\dot{y}B_z - \dot{z}B_y\right)\right]$$

$$= q\left[E_x + (\vec{v}\times\vec{B})_x\right]$$

In this approach it appears miraculous! A better derivation uses the identities in the ~~exer~~ exercise.

Comment 3

If there a dissipative parts of ~~of~~ the force, like friction, then one could write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

includes conservative part of the force

dissipative part

Example $\quad F_{fx} = -k_x v_x$ etc.

Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2} \sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2 \right)$$

$$F_{fx_i} = -\frac{\partial \mathcal{F}}{\partial v_{x_i}} \quad \text{or} \quad F_f = -\nabla_v \mathcal{F}$$

$$Q_j = \sum_i \vec{F}_{fi} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_{\vec{v}_i} \mathcal{F} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\sum_i \vec{\nabla}_{\vec{v}_i} \mathcal{F} \; \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$