Hamilton–Jacobi Theory

Time evolution is a canonical transformation

$$q = q(q_0, p_0, t)$$
$$p = p(q_0, p_0, t)$$

Final positions & momenta in terms of initial ones.

If we could go from $(q, p) \rightarrow (Q = q_0, P = p_0)$
or some func of q_0, p_0

then $\dot{Q} = 0$, $\dot{P} = 0$

$$0 = \dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad 0 = \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Having $K = 0$ achieves that!

If there is a generating function of the second kind $F_2 = S(q, P, t)$, then

$$K = H(q, p, t) + \frac{\partial S}{\partial t} = 0$$

$$p_i = \frac{\partial S}{\partial q_i}$$

So we have

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$$

Hamilton - Jacobi equation!

If H is independent of time

$$H\left(q, \frac{\partial S}{\partial q}\right) = \alpha = -\frac{\partial S}{\partial t}$$

So, try $S(q, P, t) = W(q, P) - \alpha t$

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha$$

$S \to$ Hamiton's principal function

$W \to$ Hamilton's characteristic function

In general $S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t)$

Constant momenta $P_1, \ldots, P_n$

$$Q_i = \beta_i = \frac{\partial S(q, P, t)}{\partial P_i} = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}$$

$$p_i = \frac{\partial S(q_i, \alpha, t)}{\partial q_i}$$

Use these to solve $q(\alpha, \beta, t)$
$p(\alpha, \beta, t)$.

For time-indep't case: $\alpha, \alpha_1, \ldots, \alpha_n$ are related, Any n of them could be used as indep't.

1) Easy example: Free particle

$$H = \frac{\vec{p}^2}{2m}; \quad S = W(x, y, z, P_x, P_y, P_z) - \alpha t$$

$$\frac{(\partial_x W)^2 + (\partial_y W)^2 + (\partial_z W)^2}{2m} = \alpha$$

$W = \vec{P} \cdot \vec{x}$ with $\frac{P^2}{2m} = \alpha$

Of course $\vec{P} = \vec{p}$

$$S = \vec{p}\cdot\vec{x} - \frac{p^2}{2m}t$$

$$Q_x = \frac{\partial S}{\partial p_x} = x - \frac{p_x}{m}t$$

$$Q_x = x(t) - v_x t = x(0)$$

2) Next Easy example: Harmonic Oscillator

Use characteristic function $W(q)$

$$H(q) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} = E$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{m\omega^2 q^2}{2} + \frac{\partial S}{\partial t} = 0$$

or

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{m\omega^2 q^2}{2} = \alpha$$

$$W = \int dq \sqrt{2m\alpha - m^2\omega^2 q^2}$$

and

$$S = \int dq \sqrt{2m\alpha - m^2\omega^2 q^2} - \alpha t$$

Now, we could either use P or use α.

Let's take $P = \alpha$

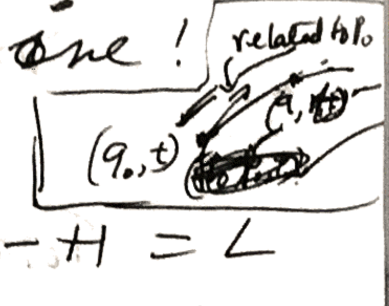
$$Q = \beta' = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} = \int \frac{m\,dq}{\sqrt{2m\alpha - m^2\omega^2 q^2}} - t$$

$$= \frac{1}{\omega} \sin^{-1} \frac{m\omega^2 q}{2\alpha} - t$$

$$q = \frac{2\alpha}{m\omega^2} \sin \omega(t + \beta')$$

Solution of harmonic osc. once more!

In general, solving such first order equation requires finding characteristic curves, which in turn is related to finding solutions of Hamilton's eqns, so we a back to square one!



Before we go on:

$$\dot{S} = \frac{\partial S}{\partial q_i} \dot{q}_i + \underbrace{\frac{\partial S}{\partial P_i} \dot{P}_i}_{0} + \frac{\partial S}{\partial t} = p_i \dot{q}_i - H = L$$

So $S \sim \int L\,dt$ on paths

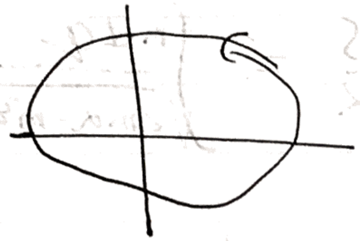
For $\frac{\partial S}{\partial t} = \alpha = H$.

$\dot{W} = p_i \dot{q}_i$

$W = \int p_i\,dq_i$ along paths

Action-Angle variables

Consider a one variable periodic motion
with $H = H(q, p)$



Consider the Variable $J(t) = \oint p\,dq$

J is obviously a constant. The orbits are constant energy curves. So $H = H(J)$

Use a characteristic function

$$F_2 = W = W(q, J) \quad , \quad \frac{\partial W}{\partial q} = p$$

The conjugate variable to [redacted] J,

$$w = \frac{\partial W}{\partial J}$$

Since $H = H(J)$ $\qquad \dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J)$

constant on the trajectory.

Total change of w through the period

$$\Delta w = \oint \frac{\partial w}{\partial q}\,dq = \oint \frac{\partial^2 W(q,J)}{\partial q\,\partial J}\,dq = \frac{\partial}{\partial J}\oint \frac{\partial W}{\partial q}\,dq$$

$$= \frac{\partial}{\partial J}\oint p\,dq = \frac{\partial J}{\partial J} = 1$$

So $\Delta w = 1 = \nu \tau$ time period

w (or $2\pi w$) is like an angle variable

One more, back to harmonic Osc.

$$J = \oint p\,dq = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\,dq$$

Substitute $q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\theta$

$$J = \sqrt{2m\alpha} \times \sqrt{\frac{2\alpha}{m\omega^2}} \oint \cos\theta \, d(\sin\theta)$$

$$= \frac{2\alpha}{\omega}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{2\pi\alpha}{\omega}$$

$\alpha = H = \frac{J\omega}{2\pi}$ $\qquad$ $\nu = \frac{\partial H}{\partial J} = \frac{\omega}{2\pi}$

$$w = \frac{\omega t + \beta}{2\pi}$$

We already saw $W = \int dq \sqrt{2m\alpha - m^2\omega^2 q^2}$

$$= \int dq \sqrt{\frac{mJ\omega}{\pi} - m^2\omega^2 q^2}$$

$$w = \frac{\partial W}{\partial J} = \frac{m\omega}{2\pi} \int \frac{dq}{\sqrt{m\omega\left(\frac{J}{\pi} - m\omega q^2\right)}} = \frac{1}{2\pi} \int \frac{dq}{\sqrt{\frac{J}{\pi m\omega} - q^2}}$$

$$= \frac{1}{2\pi} \sin^{-1} \frac{q}{\sqrt{\frac{J}{\pi m\omega}}}$$

$$q = \sqrt{\frac{J}{\pi m\omega}} \sin 2\pi w$$

For a completely separable system

$$W(q, \alpha_1 \ldots \alpha_n) = \sum_i W_i(q_i, \alpha_1 \ldots \alpha_n)$$

Action var. $\rightarrow J_i = \oint p_i \, dq_i$ (no summ^{n} convention)

$$= \oint \frac{\partial W_i}{\partial q_i} dq_i$$

Angle var $\rightarrow w_i = \frac{\partial W}{\partial J_i} = \sum_j \frac{\partial W_j}{\partial J_i}(q_j, J_1, \ldots, J_n)$

$$\dot{w}_i = \frac{\partial H(J_1, \dots, J_n)}{\partial J_i} = \nu_i$$

n-dim Torus.

An interesting example is motion due to central force in a plane

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

$l = p_\theta$ is a constant of motion

$$\oint p_\theta \, d\theta = 2\pi p_\theta = J_\theta$$

$H = \alpha$

$$J_r = \oint \sqrt{2m\left(\alpha - V(r) - \frac{J_\theta^2}{4\pi^2 r^2}\right)} \, dr$$

In principle, this relation could be used to express $\alpha = H(J_\theta, J_r)$

$$W = \int p_\theta \, d\theta + \int p_r \, dr$$

$$= \frac{1}{2\pi} J_\theta \theta + \int \sqrt{2m\left(\alpha - V(r) - \frac{J_\theta^2}{4\pi^2 r^2}\right)} \, dr$$

$$= \frac{1}{2\pi} J_\theta \theta + \int \sqrt{2m\left(H(J_r, J_\theta) - V(r) - \frac{J_\theta^2}{4\pi^2 r^2}\right)} \, dr$$

$$w_r \propto \int \frac{dr}{\sqrt{2m\left(\alpha - V(r) - \frac{J_\theta^2}{4\pi^2 r^2}\right)}} \propto t$$

$2\pi w_\theta = \theta -$ additional term that are r dependent...

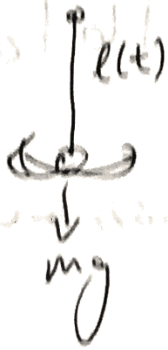
does not change uniformly in time except for circular orbits

Compensation from r dependent terms.

Usefulness in perturbation theory

$$q_k = \sum_{j_1 \dots j_n} a_{j_1 \dots j_n} e^{2\pi i (j_1 w_1 + \dots + j_n w_n)}$$

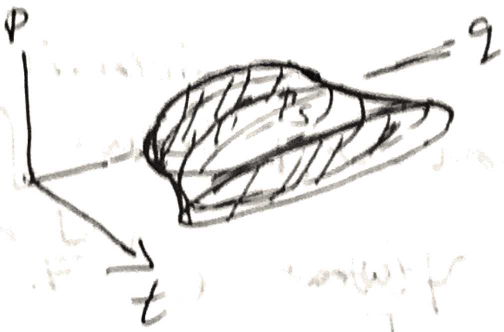
Adiabatic Invariants



Pendulum with slowly varying length

How does its amplitude evolve in time?

$H(q,p,t)$



$$\int_{\partial V} \Omega = 0$$

$$\Omega = d(p\,dq - H\,dt)$$

$$\left(\oint p\,dq\right)_f - \left(\oint p\,dq\right)_i$$

$$+ \int_S \Omega$$

$\int \Omega = 0 \iff$ eqn of motion.

strip between trajectories

$\Rightarrow \left(\oint p\,dq\right)_f = \left(\oint p\,dq\right)_i$, Since $J_i = \int_{T_i} p\,dq \approx \left(\oint p\,dq\right)_i$

and $J_f = \int_{T_f} p\,dq \approx \left(\oint p\,dq\right)_f$

$J_i \simeq J_f$

That implies $\frac{E}{\omega}$ is fixed, for harmonic oscillators.

Used in Bohr-Sommerfeld quantization

$$J = nh$$

This is because semiclassical wave functions are like

$$\psi_J(w) \sim e^{\frac{iJw}{\hbar}} = e^{\frac{2\pi i J w}{h}}$$

$w \to w+1$ is the same point in phase space.

$$\Rightarrow \quad J = nh$$